Combinatorial Structure of Arrangements

$\mathcal{A} = \{H_1, \ldots, H_n\}$ hyp. arr. in $V = \mathbb{K}^\ell$, $\mathbb{K}$-field

$\alpha_i \in V^*$ w/ $\ker \alpha_i = H_i$.

Properties of minimal ~~In~~dependent sets (Whitney, Maclane)

Let $\mathcal{C}$ = set of minimal dep. subsets of $\{\alpha_1, \ldots, \alpha_n\} \subseteq V^*$ "circuits"

$\boxed{C \in \mathcal{C} \iff \sum_{i \in C} c_i \alpha_i = 0 \text{ w/ } c_i \neq 0 \ \forall i}$

(i) $\emptyset \notin \mathcal{C}$

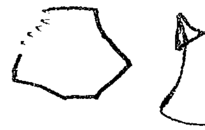
(ii) $A \in \mathcal{C}$ and $B \supsetneq A$ then $B \notin \mathcal{C}$

(iii) if $A, B \in \mathcal{C}$ and $e \in A \cap B$ then $\exists\, C \in \mathcal{C}$ with $C \subseteq A \cup B - \{e\}$

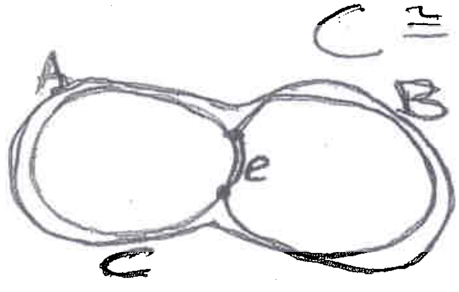
Def: A matroid w/ ground set $[n] = \{1, \ldots, n\}$ or E is (determined by) a set $\mathcal{C} \subseteq 2^{[n]}$ or 2^E satisfying (i)–(iii)

Ex: Let Γ be a finite graph, $E = E(\Gamma)$ the set of edges.

Let $\mathcal{C} = \{C \subseteq E \mid C \text{ is a cycle (or circuit)}\}$

$C \cong$  Graphic Matroid

Here's (iii)



<u>Graphic Arrangements</u> Γ = graph w/o loops or multiple edges

$$\mathcal{A}_\Gamma = \{ H_{ij} \mid ij \in E(\Gamma) \} \quad H_{ij} = \ker(x_i - x_j)$$

circuit in $\mathcal{A}_\Gamma \Leftrightarrow$ circuit in Γ

<u>Note:</u> Graphic arr's $\longleftrightarrow$ subarr's of the braid ar

$\updownarrow$

Graphic matroids

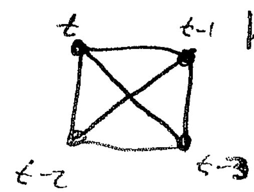
<u>chromatic polynomial</u>

$\chi_\Gamma(n)$ = # of colorings of vertices of Γ w/ n colors and adjacent vertizes having different colors

change of variables

$\mathrm{Poin}(\mathcal{A}_\Gamma, t)$

<u>Ex:</u> Γ



K_4

$\mathcal{A}_\Gamma$ = braid arr. $= \mathcal{A}_4$

$$\chi_\Gamma(t) = t(t-1)(t-2)(t-3)$$

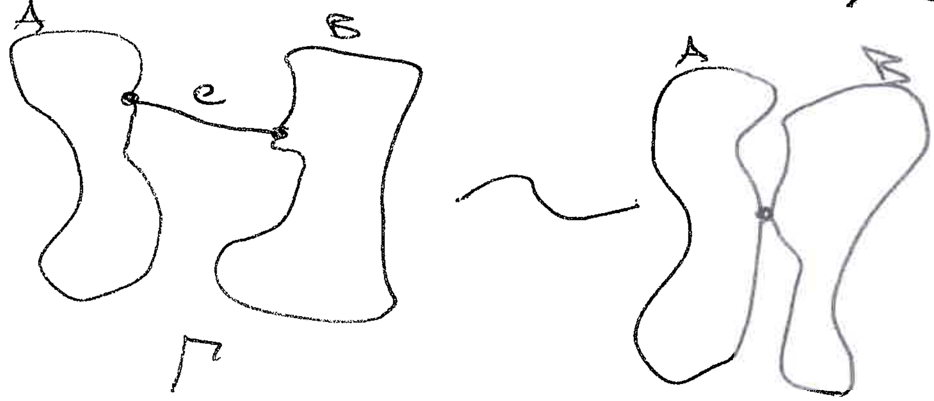
$$\mathrm{Poin}(\mathcal{A}_4, t) = (1+0t)(1+t)(1+2t)(1+3t)$$

$$\mathrm{Poin}(\mathcal{A}_\Gamma, t) = (-t)^{r(\Gamma)} \chi_\Gamma\left(-\tfrac{1}{t}\right)$$

deletion of $e=ij$ from Γ $= \Gamma - e$

$\leftrightarrow$ deletion of H_{ij} from $\mathcal{A}_\Gamma$, $\mathcal{A}'_\Gamma$

contraction of e in Γ $= \Gamma/e$



$\leftrightarrow$ contraction or restriction $\mathcal{A}''_\Gamma$ of H_{ij} in $\mathcal{A}_\Gamma$ in $(\mathcal{A}_\Gamma, t)$

<u>Thm</u>: $\chi_\Gamma(n) = \chi_{\Gamma - e}(n) - \chi_{\Gamma/e}(n)$

<u>Pf</u>: "different = all − same"

<u>Cor</u>: $\mathrm{Poin}(\mathcal{A}_\Gamma, t) = \mathrm{Poin}(\mathcal{A}'_\Gamma, t) + t\,\mathrm{Poin}(\mathcal{A}''_\Gamma, t)$

<u>Note</u>: This follows from $0 \to A' \to A \to A''(-1) \to 0$

<u>Remark</u>: $\exists$ many equivalent axiomatizations of matroids in terms of:

- dependent sets
- indep. sets
- maximal indep. sets
- closed sets
- closure operator
- "hyperplanes" or "bonds"

$\Big\}$ cryptomorphisms

or 30 others

Closed sets Let G be a matroid on $[n]$. $S \subseteq [n]$

The closure, $\overline{S}$, of S is defined by

$$i \in \overline{S} \iff \exists \text{ circuit } C \in G \text{ with } i \in C \text{ and } |C \cap S| \geq |C|-1$$

S is closed iff $\overline{S} = S$

closed sets $\equiv$ "flats"

The collection of closed sets $L(G)$ is a geom. la

If G is the underlying matroid of $\mathcal{A}$ then $L(G) \cong L(\mathcal{A})$.

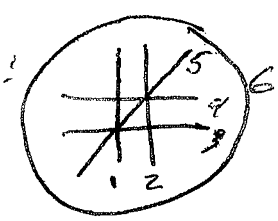
Pictures

hyp. arr.'s in $\underset{\shortparallel\, V}{\mathbb{K}^\ell}$ $\longleftrightarrow$ point config.'s in $\mathbb{P}(V^*)$.

$\{H_1, \ldots, H_n\} \longleftrightarrow \{\alpha_1, \ldots, \alpha_n\}$

$\uparrow$ normal lines

Ex:



$H_1 = X$ $\alpha_1 = [1\ 0\ 0]$

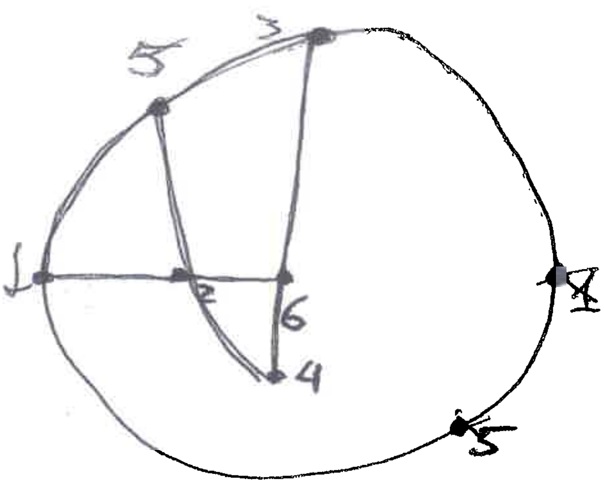
$H_2 = X - Z$ $\alpha_2 = [1\ 0\ -1]$

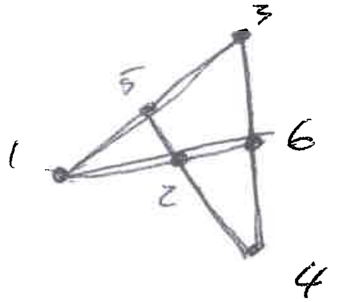
$H_3 = Y$ $\alpha_3 = [0\ 1\ 0]$

$H_4 = Y - Z$ $\alpha_4 = [0\ 1\ -1]$

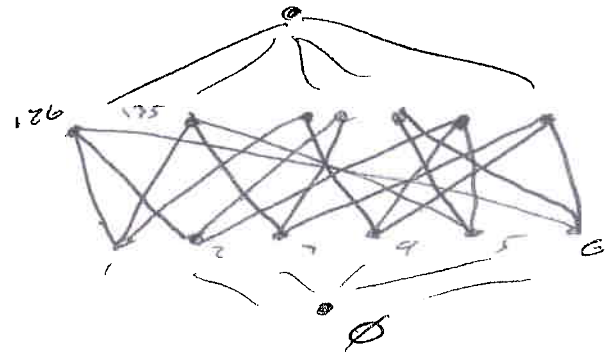
$H_5 = X - Y$ $\alpha_5 = [1\ -1\ 0]$

$H_6 = Z$ $\alpha_6 = [0\ 0\ 1]$



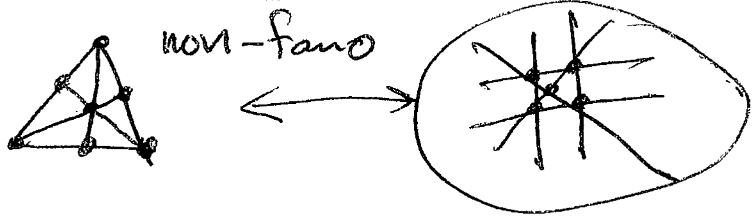


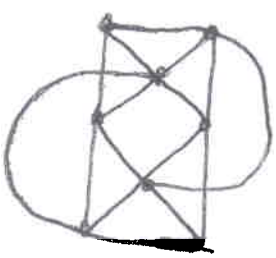
135 is a flat



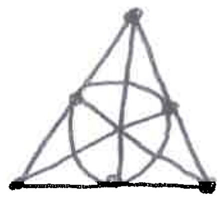
Other Examples of Matroids

non-fano





MacLane



fano

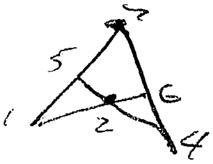
Realization Spaces

Problem: Given a matroid G and a field $\mathbb{K}$, construct a parametrization of the set of all labeled proj. pt config.'s in $\mathbb{P}(\mathbb{K}^{\ell})$ w/ matroid G, up to proj. equiv.

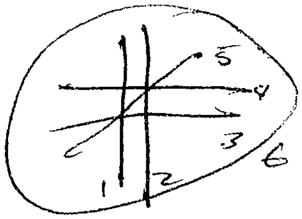
Ex 1 "the braid arr. $\mathcal{A}_4$ is proj. unique"

Method: choose a proj. basis B in G

B has $B = \{i_1, i_2, i_3, i_4\}$ with each triple $\subseteq B$ independent.

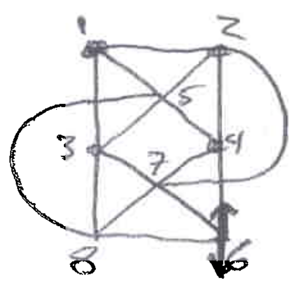
Ex: 

$B = 1234$



Any realization of G is proj. eq. to a unique realization of the form

$$\begin{array}{r l}
1 & \\
2 & \\
3 & \\
4 & \\
(1\vee 2)\wedge(3\vee 4)\cancel{5} & \\
= 6 &
\end{array}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Ex: 

$$\begin{array}{r}
1 \\
2 \\
3 \\
4 \\
(1\vee 4)\wedge(2\vee 3) = 5 \\
(3\vee 6)\wedge(1\vee 2) = 6 \\
= 7 \\
(1\vee 3)\wedge(4\vee 7) \text{ and } (5\vee 6) = 8
\end{array}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & t & 1 \\ 1 & t & 0 \\ t-1 & 0 & t \end{bmatrix}$$

$t \neq 1$

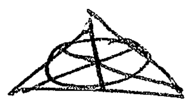
$$\det(568) = 0 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & t & 1 \\ t-1 & 0 & t \end{vmatrix} = t^2 + t + 1$$

So, $t = \omega$, ... where $\omega^3 =$ ...

Conclusions:

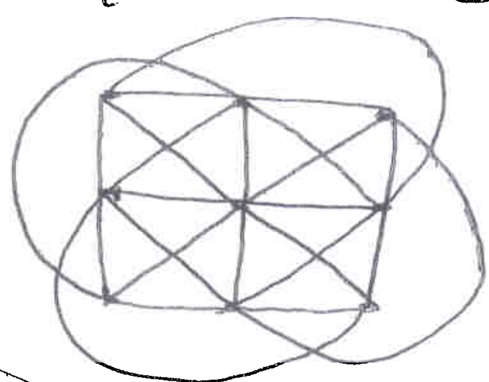
- $\exists$ 2 inequivalent realizations over $\mathbb{C}$
- $\nexists$ realizations over $\mathbb{R}$
- $\exists$ realizations over $\mathbb{K}$ if $\mathrm{char}(\mathbb{K})=3$

Exercises

(1) Show  is realizable only over fields of $\mathrm{char}(\mathbb{K})=2$.

(2) Is the arr. $Q(\mathcal{A})=(x^m-y^m)(x^m-z^m)(y^m-z^m)$ proj. unique over $\mathbb{C}$

$m=3$



Ref.

Bokowski & Sturmfels Comp. Synth. Geom.