Lecture 8 Falk (48)

Combinatorics & Topology of the Complement: Topological Classification

$\mathcal{A}$: arr. in $\mathbb{C}^\ell$, $M = M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup \mathcal{A}$

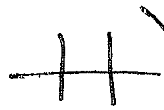
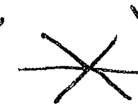
$L = \cap$ - lattice

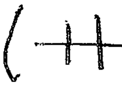
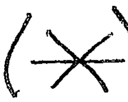
Thm: (i) $H^*(M) \cong A(\mathcal{A})$ as rings

(ii) $A(\mathcal{A})$ is uniquely determined by $L(\mathcal{A})$

Question 1: Is the homotopy type or homeomorphism type of M determined by $L(\mathcal{A})$?

Answer: No

Ex: $\pi_1($  $) \cong F_2 \times F_1 \cong \pi_1($  $)$

In fact, $M($  $) \cong M($  $)$

$\mathcal{A}_1 =$  $\mathcal{A}_2 =$ 

$Q_1 = (x+1)(x-1)y$ $Q_2 = (x+y)(x-y)y$

"$c\mathcal{A}_1$:  $c\mathcal{A}_2$: 

$cQ_1 = (x+z)(x-z)yz$ $cQ_2 = (x+y)(x-y)yz$

$$M(\mathcal{A}_1) = M(d(c\mathcal{A}_1)) = \underbrace{M(c\mathcal{A}_1)/\mathbb{C}^*}_{\text{projective image}} \cong M(c\mathcal{A}_2)/\mathbb{C}^*$$

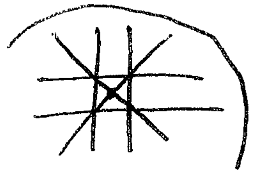
$$= M(d(c\mathcal{A}_2)) = M(\mathcal{A}_2)$$

Question 2: Are there central arr.'s for which $M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$ but $L(\mathcal{A}_1) \not\cong L(\mathcal{A}_2)$?

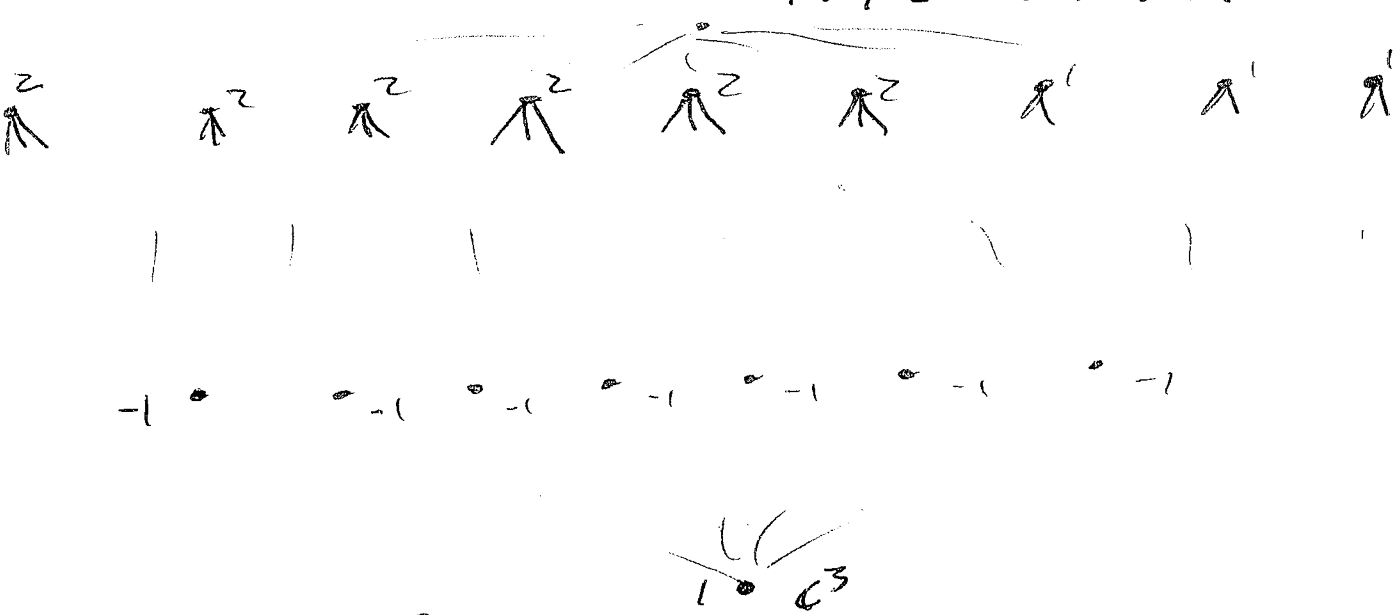
OR: Are there central arr.'s for which the $A(\mathcal{A}_1) \cong A(\mathcal{A}_2)$ but $L(\mathcal{A}_1) \not\cong L(\mathcal{A}_2)$?

Answer: Yes. So, instead we ask the following: What combinatorial features of L can be extracted from A?

Observation 1 $A(\mathcal{A})$ determines $\mathrm{Poin}(L,t) = \sum_{x \in L} |\mu(x)| t$

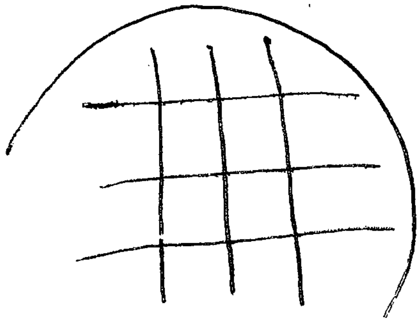
Ex:  is inductively free (1, 3, 3)

$$\mathrm{Poin}(L,t) = (1+t)(1+3t)^2$$
$$= 1 + 7t + 15t^2 + 9t^3$$



The coeff.'s of $\mathrm{Poin}(L,t)$ are called "Whitney numbers of L of the 2^{nd} kind"

Ex:



supersolvable

$\mathrm{Poin}(L,t) = (1+t)(1+3t)^2$

The O.S. alg.'s are not iso. since one is supersolv. and the other isn't.

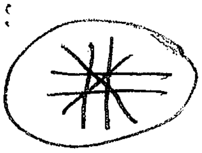
Construction: An invariant of the ring structure of $A(L)$.

$A = E/I$, $A^1 \cong E^1$, $I^2 \cong \ker(\underset{E^2}{\overset{\shortparallel}{\Lambda^2(A^1)}} \to A^2)$

$$\begin{aligned} \Delta : E^1 \otimes I^2 &\longrightarrow E^3 = \Lambda^3(A^1) \\ x \otimes r &\longmapsto x \wedge r \end{aligned}$$

Def: $\varphi_3 = \dim(\ker \Delta)$

Ex:



$\varphi_3 = 17$

$\varphi_3 = 12$

$I^2 = \langle de_S \mid S \text{ dependent}, |S| = 3 \rangle$

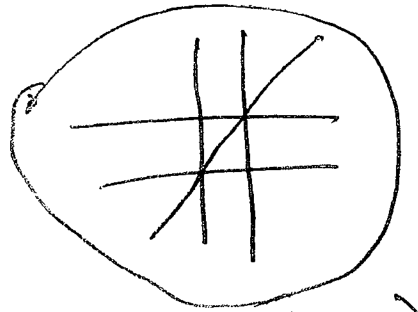
$= \langle de_{ijk} = (e_i - e_j) \wedge (e_i - e_k) \mid \{i,j,k\} \text{ dependent} \rangle$

$\Downarrow$

$(e_i - e_j) \otimes de_{ijk} \in \ker \Delta$

$(e_i - e_k) \otimes de_{ijk} \in \ker \Delta$

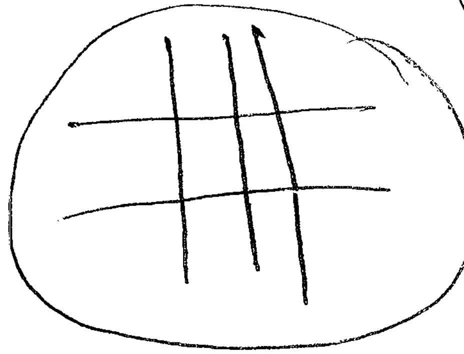
Ex



both super-solvable

$\exp = (1,2,3)$

φ_3 is the same $=10$



$\exp = (1,2,3)$

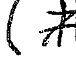
Thm (Sullivan, Morgan: rational htpy theory)

$\varphi_3 =$ rank of the 3rd factor in the lower central series of π_1

$$\pi = \pi^1,\ [\pi,\pi] = \pi^2,\ \ldots,\ [\pi,\pi^k] = \pi^{k+1}$$

$$\varphi_3 = \operatorname{rank}\left(\pi^3/\pi^4\right)$$

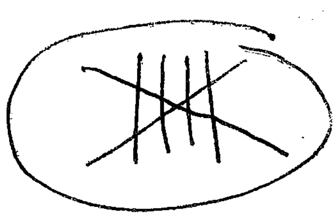
Thm (Falk, Randell) If $\mathcal{A}$ is supersolvable, then ranks of factors in the L.C.S. of π_1 are determined by the exponents.

Ex: $\pi_1($  $) = P_4$

$\pi_1($  $) = F_3 \times F_2 \times F_1$.

Question: Is $A(\mathcal{A}_1) \cong A(\mathcal{A}_2)$? Answer: No

Exercise: $\mathcal{A}$



show $\operatorname{Pom}(L(\mathcal{A}), t)$ splits over $\mathbb{Z}$ but $\mathcal{A}$ is not free.

Question: Are there arr.'s $\mathcal{A}_1$ and $\mathcal{A}_2$ for which $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$ but (i) $M(\mathcal{A}_1) \not\cong M(\mathcal{A}_2)$? Yes

(ii) $\pi_1(M(\mathcal{A}_1)) \not\cong \pi_1(M(\mathcal{A}_2))$? Yes

Projective equivalence

$$\mathcal{A} = \{H_1, \dots, H_n\} \text{ central} \quad x_i \in V^* \text{ w/ basis } x_1, \dots, x_\ell \quad \text{s.t. } H_i = \ker(\alpha_i)$$

$$B = B(\mathcal{A}) = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}^{n \times \ell}$$

Then $\mathcal{A}_1$ is projectively equivalent to $\mathcal{A}_2$ iff $\exists$ $\ell \times \ell$ non-singular matrix C and $n \times n$ non-singular diagonal matrix D s.t.

$$B(\mathcal{A}_2) = D\, B(\mathcal{A}_1)\, C.$$

If $\mathcal{A}_1$ is projectively equi. to $\mathcal{A}_2$ then

$$M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$$

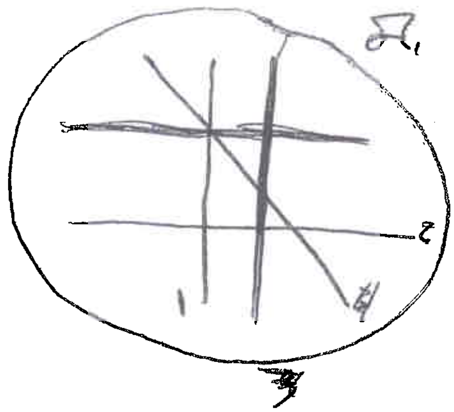
Normal forms (for central arr. of rank 3)

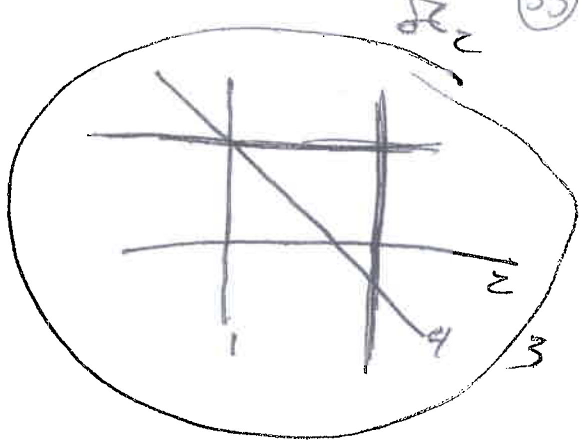
Assume H_1, H_2, H_3, H_4 are in "general position" (so it will look like ) (none $\{H_i, H_j, H_k\}$ are dependent)

Then $\mathcal{A}$ is projectively equ. to a unique arr. $\mathcal{A}_0$ with

$$B(\mathcal{A}_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Ex





$\mathcal{A}_1$ is not proj. eq. to $\mathcal{A}_2$

but $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$.

Is $M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$?

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Randall's Lattice-isotopy theorem

Suppose $\mathcal{A}_t = \{H_1(t), \dots, H_n(t)\}$

$0 \le t \le 1$ is a family of arr.'s

satisfying $\operatorname{codim}\left(\bigcap_{i \in S} H_i(t)\right)$ is constant

for every S.

Then, $\exists$ a continuous change of variables carrying $(\mathbb{C}^\ell, \cup \mathcal{A}_0)$ to $(\mathbb{C}^\ell, \cup \mathcal{A}_1)$ and $M(\mathcal{A}_0)$ is diffeomorphi to $M(\mathcal{A}_1)$.