Hyperplane Arrangements Workshop

Lecture 1 Falk

§1 Let $\mathcal{A} = \{L_1, \ldots, L_n\}$ — arrangement of lines through 0 in $\mathbb{C}^2$, $M = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} L_i$.

Then $M \cong (M \cap S^3) \times (0, \infty) \simeq M \cap S^3$

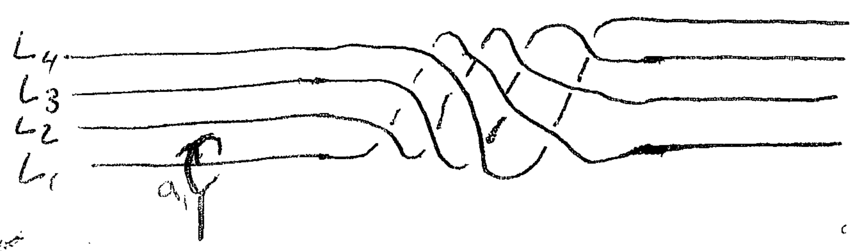
$= S^3 \setminus \bigcup_{i=1}^{n} (L_i \cap S^3)$. The lines determine points in $\mathbb{CP}^1 \cong S^2$.

$L_i \cap S^3$ is a fiber of $h: S^3 \to \mathbb{CP}^1 = S^2$

$\{L_i \cap S^3\}$ are $(1,1)$ curves on tori in S^3

So, $\bigcup_{i=1}^{n} (L_i \cap S^3)$ is isotopic to an (n,n)-torus link $L_{n,n}$ in S^3 $M \simeq S^3 - L_{n,n}$

ex: $n = 4$



Exercise: Calculate the Wirtinger presentation of $\pi_1(S^3 - L_{n,n}) = \langle a_1, \ldots, a_n \mid a_n a_{n-1} \cdots a_2 a_1 = a_{n-1} a_{n-2} \cdots a_2 a_1 a_n$

$= a_{n-2} \cdots a_1 a_n a_{n-1}$

$\vdots$

$= a_1 a_n a_{n-1} \cdots a_2 \rangle$

denoted $[a_n, \ldots, a_n]$ The Randell relations

§2 | $S^3 - L \longrightarrow \mathbb{C}P^1 - \{\text{one pt.}\} \cong D^2$

is a trivial bundle

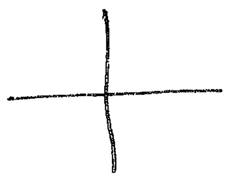
So, $S^3 \setminus L_{n,n} \longrightarrow \mathbb{C}P^1 - \{n \text{ pts}\}$ also trivial

So, $S^3 \setminus L_{n,n} \cong S^1 \times (S^2 \setminus n \text{ pts}) = D^2 - (n-1 \text{ pts})$

Then $\pi_1(S^3 - L_{n,n}) \cong F_1 \times F_{n-1}$

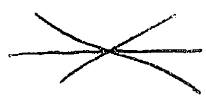
More precisely, $S^3 - L_{n,n} \simeq S^1 \times (\overset{n-1}{\vee} S^1)$

Ex: $n = 2$



$M \simeq S^1 \times S^1$

$n = 3$



$M \simeq S^1 \times (S^1 \vee S^1)$

= the union of two that meet along a (1,1)-curve

The Randell relations are equivalent to

$[a_n \cdots a_1, a_i] = 1 \quad \forall i$

§3 | Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ an arrangement of linear* hyperplanes in $\mathbb{C}^\ell$

* $\mathcal{A}$ is central

$$\pi : \mathbb{C}^\ell \setminus \{0\} \longrightarrow \mathbb{C}P^{\ell-1}$$

$$\pi|_{\mathbb{C}^\ell \setminus H_1} : \mathbb{C}^\ell \setminus H_1 \longrightarrow \mathbb{C}P^{\ell-1} \setminus \overset{\overset{\pi(H_1)}{\shortparallel}}{\bar{H}_1} \cong \mathbb{C}^\ell$$

is a trivial bundle.

So, $\pi|_M : M \to \mathbb{CP}^\ell \setminus \bigcup_{i=1}^{n} H_i$ is trivial,

$$M \cong \mathbb{C}^* \times \left(\mathbb{CP}^{\ell-1} - \bigcup_{i=1}^{n} \overline{H_i}\right) \cong \mathbb{C}^{\ell-1} - \bigcup_{i=2}^{n} \left(\overline{H_i} \cap \mathbb{C}^{\ell-1}\right)$$

$\{\overline{H_i} \cap \mathbb{C}^{\ell-1} \mid 2 \leq i \leq n\}$ is an arrangement of $n-1$ affine hyperplanes in $\mathbb{C}^{\ell-1}$ called the <u>decone</u> of $\mathcal{A}$ written $d\mathcal{A}$.

<u>Thm</u>: $M(\mathcal{A}) \cong \mathbb{C}^* \times M(d\mathcal{A})$

Cor.: $\chi(M(\mathcal{A})) = 0$

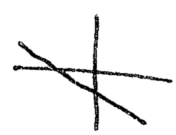
<u>Ex</u>: $\ell = 3$ $\mathcal{A}$ given by linear factors of

$$Q(\mathcal{A}) = x\,y\,z\,(x+y+z)$$

$H_1 = \{z = 0\}$

$d\mathcal{A}$ given by $Q(d\mathcal{A}) = xy(x+y+1)$

is the projective image



The process is reversible:
given an aff. arr. $\overline{\mathcal{A}}$ in $\mathbb{C}^{\ell-1}$
$\exists$ a central arr. $\mathcal{A}$ in $\mathbb{C}^\ell$ w/ $\overline{\mathcal{A}} = d\mathcal{A}$
$\mathcal{A}$ is called the <u>cone</u> of $\overline{\mathcal{A}}$, write $c\overline{\mathcal{A}}$.

{4} $\mathcal{A}$ is a central arr. in $\mathbb{C}^\ell$

$H_i = \ker\left(\alpha_i : \mathbb{C}^\ell \xrightarrow{\text{linear}} \mathbb{C}\right)$. Say $\{H_{i_1}, \ldots, H_{i_p}\}$ is <u>independent</u> if $\{\alpha_{i_1}, \ldots, \alpha_{i_p}\}$ is linearly independent in $(\mathbb{C}^\ell)^*$

If $\mathcal{B} \subseteq \mathcal{A}$ then $M(\mathcal{B}) \overset{i}{\supseteq} M(\mathcal{A})$ so

$$H^*(M(\mathcal{B})) \xrightarrow{i_*} H^*(M(\mathcal{A}))$$

If $\mathcal{B} = \{H_{i_1}, \dots, H_{i_p}\}$ is independent then the linear map $\mathbb{C}^\ell \to \mathbb{C}^p$

$$x \longmapsto (\varphi_{i_1}(x), \dots, \varphi_{i_p}(x)) \quad \text{has kernel} = \bigcap_{k=1}^{p} H_{i_k} = X$$

and restricts to a homotopy equivalence

$$M(\mathcal{B}) \longrightarrow (\mathbb{C}^*)^p$$

$$(M(\mathcal{B}) \simeq (\mathbb{C}^*)^p \times X) \qquad \text{so, we get a}$$

homom.
$$\Lambda(e_1, \dots, e_p) \cong H^*((\mathbb{C}^*)^p) \longrightarrow H^*(M(\mathcal{A}))$$

The images of these maps span $H^*(M(\mathcal{A}))$

<u>Cor.</u> $H^1(M)$ generates $H^*(M)$

<u>Cor.</u> (Randell) The Hurewicz homom.

$$\pi_k(M) \longrightarrow H_k(M) \quad \text{is trivial } \forall k \geq 2.$$

<u>Proof</u>: Exercise

§5 Randell's model for complexified real affine 2-arr.

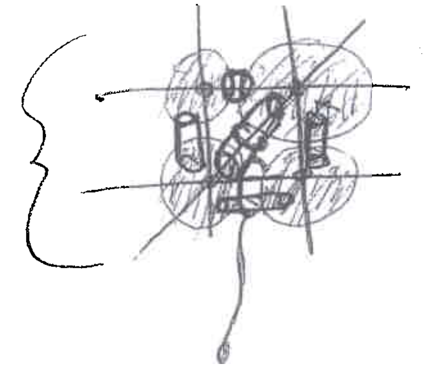
$\mathcal{A}$ is complexified real if the φ_i have real coeff

$\mathcal{A}$ = complexified real central 3-arr.

$$\Rightarrow M(\mathcal{A}) \simeq \mathbb{C}^* \times M(d\mathcal{A}) = \mathbb{C}^* \times \overline{M}$$

complexified real affine 2-arr.

ex:

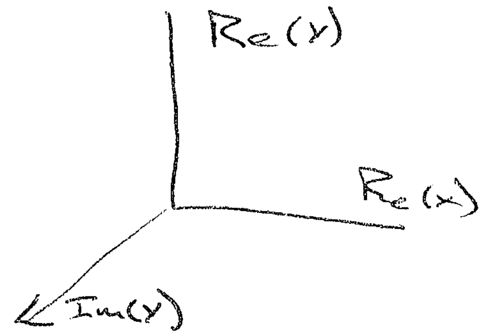
Γ 

For each intersection pt. X

let $B_X = \overline{M} \cap B_{\mathbb{C}^2}(X, \epsilon)$

$B_X \simeq S^3 - L_{m,m}$ where m = mult. of X

$$\overline{M} \simeq \overline{M} \cap (\mathbb{R}^3 \times \overbrace{[-\delta, \delta]}^{\mathrm{Im}(y)})$$

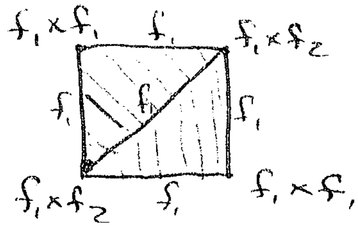


$$= (\mathbb{R}^3 - \Gamma) \cup \bigcup_X B_X$$

and $(\mathbb{R}^3 - \Gamma) \cap B_X = \mathbb{R}^3 - X$.

This yields the "tinker-toy" model of $\overline{M}$,

$\overline{M}$ = total space of a 2-complex of aspherical spaces



Huisman model for projective 2-arr.

$\mathcal{A}$ = central complexified real 3-arr.
$= \{H_1, \dots, H_n\}$

"Model for $\overline{M} = \mathbb{CP}^2 - \bigcup_{i=1}^{n} \overline{H_i}$.

Each $\overline{H_i}$ ~~is a~~ determines a line in ~~CP~~ $\mathbb{RP}^2$

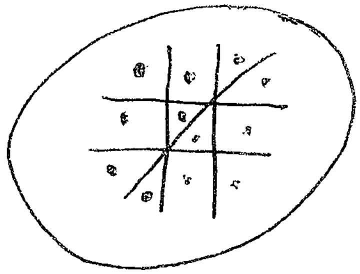
Let $\overline{P_i}$ = the point in $\mathbb{RP}^2$ dual to $\overline{H_i}$

Let $\pi : S^2 \to \mathbb{RP}^2$.

Let $\{P_1, \dots, P_{2n}\}$ be the preimage of $\{\overline{P_1}, \dots, \overline{P_n}\}$

Choose one pt q_j in each component of

$\mathbb{R}P^2 - \bigcup_{i=1}^{n} H_i$



Each q_j determines a dual line in $\mathbb{R}P^2$, hence a great circle in S^2, call it C_j.

Thm: $M \simeq$ ~~$(S^2 - \{P_1, \dots, P_{2n}\})$~~ (homotopy equ.) the space obtained from $S^2 - \{P_1, \dots, P_{2n}\}$ by attaching disks to the C_j's

Exercise ① Find the Huisman model for #

② Show the Huisman model has the correct fundamental grp.