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MATH 4565 **TOPOLOGY** **Spring 2010**

Degree

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1. The path-lifting lemma

We start with a basic technical lemma that will allow us to define the degree of a continuous map from the circle S^1 to itself.

Let $e\colon \mathbb{R} \to S^1$ be the "exponential map." If we view S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$, then e is given by

$$e(t) = \exp(2\pi i t). \tag{1}$$

Or, if we view S^1 as $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, then e is given by

$$e(t) = (\cos(2\pi t), \sin(2\pi t)). \tag{2}$$

Clearly, e is a continuous surjection. Since both the functions $t \mapsto \cos(2\pi t)$ and $t \mapsto \sin(2\pi t)$ are periodic of period 1, we have

$$e(t + C) = e(t), \quad \text{for every } C \in \mathbb{Z}. \tag{3}$$

Moreover,

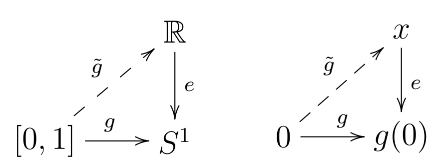
$$e(t_1) = e(t_2) \Longleftrightarrow t_1 - t_2 \in \mathbb{Z} \tag{4}$$

Finally, let $I = [0, 1]$ be the unit interval in $\mathbb{R}$, and denote by $e_0\colon I \to S^1$ the restriction of e to I. Again, e_0 is a continuous surjection—in fact, a quotient map.

Lemma 1.1 (Path-Lifting Lemma)**.** *Let $g\colon [0, 1] \to S^1$ be a continuous map, and let $x \in \mathbb{R}$ such that $e(x) = g(0)$. There is then a unique continuous map $\tilde{g}\colon [0, 1] \to \mathbb{R}$ such that*

- *$e(\tilde{g}(t)) = g(t)$, for all $t \in [0, 1]$.*
- *$\tilde{g}(0) = x$.*

A map $\tilde{g}$ as above is called a *lift* of g. Once we impose the "initial condition" $\tilde{g}(0) = x$, we get the *unique* lift of g at x. The situation is summarized in the commuting diagrams



An analogous statement holds for homotopies.

Lemma 1.2 (Homotopy-Lifting Lemma)**.** *Let $F\colon I \times I \to S^1$ be a continuous map, and let $x \in \mathbb{R}$ such that $e(x) = F(0,0)$. There is then a unique continuous map $\tilde{F}\colon I \times I \to \mathbb{R}$ such that*

- $e(\tilde{F}(t,s)) = g(t,s)$, *for all* $t, s \in I \times I$.
- $\tilde{F}(0,0) = x$.

2. The degree of a circle map

Let $f\colon S^1 \to S^1$ be a continuous map. Consider the composite

$$g = f \circ e_0\colon [0,1] \to S^1. \tag{5}$$

Clearly, g is a continuous map. Moreover, since $e_0(0) = e_0(1) = (1,0)$, we have:

$$g(0) = g(1). \tag{6}$$

By Lemma 1.1, the map g admits a lift $\tilde{g}\colon [0,1] \to \mathbb{R}$. That is to say, $e \circ \tilde{g} = g$. From (6), we get:

$$e(\tilde{g}(0)) = e(\tilde{g}(1)) \tag{7}$$

Applying (4), we obtain:

$$\tilde{g}(1) - \tilde{g}(0) \in \mathbb{Z} \tag{8}$$

This leads to the following definition

Definition 2.1. With notation as above, the *degree* of the map $f\colon S^1 \to S^1$ is the integer $\deg(f)$ given by

$$\deg(f) = \tilde{g}(1) - \tilde{g}(0).$$

We must verify that $\deg(f)$ is well-defined, i.e., does not depend on the choice of lift $\tilde{g}$ for $g = f \circ e_0$. So, suppose $\bar{g}\colon [0,1] \to \mathbb{R}$ is another lift of g. Note that

$$e(\tilde{g}(0)) = e(\bar{g}(0)) = g(0). \tag{9}$$

Thus, again by (4), we must have

$$\tilde{g}(0) - \bar{g}(0) = C, \quad \text{for some } C \in \mathbb{Z}. \tag{10}$$

Consider the map $\tilde{\bar{g}}\colon [0,1] \to \mathbb{R}$ given by

$$\tilde{\bar{g}}(t) = \bar{g}(t) + C. \tag{11}$$

We then have, by (3),

$$e(\tilde{\bar{g}}(t)) = e(\bar{g}(t) + C) = e(\bar{g}(t)) = g(t). \tag{12}$$

In other words, $\tilde{\bar{g}}$ is also a lift of g. Combining (9) and (11), we see that

$$\tilde{\bar{g}}(0) = \bar{g}(0) + C = \tilde{g}(0). \tag{13}$$

That is, the two lifts, $\tilde{g}$ and $\tilde{\bar{g}}$, agree at 0. By the uniqueness statement from Lemma 1.1, these two lifts must agree, for all $t \in [0,1]$; that is,

$$\tilde{\bar{g}} = \tilde{g}. \tag{14}$$

In view of (11), this is the same as saying

$$\tilde{g}(t) = \bar{g}(t) + C, \quad \text{for all } t \in [0,1]. \tag{15}$$

Therefore,

$$\bar{g}(1) - \bar{g}(0) = (\tilde{g}(1) - C) - (\tilde{g}(0) - C) = \tilde{g}(1) - \tilde{g}(0), \tag{16}$$

showing that we obtain the same value for $\deg(f)$, whether we use the lift $\tilde{g}$, or the lift $\bar{g}$ in Definition 2.1.

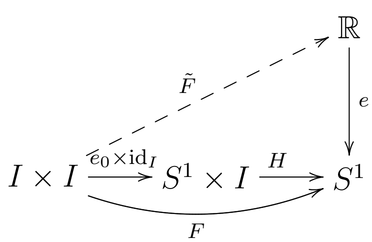
The following theorem shows that the degree of a circle map depends only on its homotopy class.

Theorem 2.2. *Let $f, g\colon S^1 \to S^1$ be two continuous maps. Suppose $f \simeq g$. Then $\deg(f) = \deg(g)$.*

Proof. Let $H\colon S^1 \times I \to S^1$ be a homotopy from f to g. That is,

$$H(z,0) = f(z) \text{ and } H(z,1) = g(z), \text{ for all } z \in S^1. \tag{17}$$

Consider the map $F\colon I \times I\colon S^1$ obtained by composing H with the map $e_0 \times \mathrm{id}_I$. By Lemma 1.2, the map F lifts to a map $\tilde{F}\colon I \times I \to \mathbb{R}$; this maps fits into the commuting diagram



Clearly, the map $t \mapsto \tilde{F}(t,0)$ is a lift of f, and likewise, the map $t \mapsto \tilde{F}(t,1)$ is a lift of g. By the definition of degree, we have

$$\deg(f) = \tilde{F}(1,0) - \tilde{F}(0,0) \tag{18}$$

$$\deg(g) = \tilde{F}(1,1) - \tilde{F}(0,1) \tag{19}$$

Now consider the continuous map $G\colon [0,1] \to \mathbb{Z}$ given by

$$G(s) = \tilde{F}(1,s) - \tilde{F}(0,s), \quad \text{for all } s \in [0,1]. \tag{20}$$

Since the interval $[0,1]$ is connected, and $\mathbb{Z}$ is discrete, the image of G must be a singleton, i.e., G is a constant function. Putting things together, we find:

$$\deg(f) = G(0) = G(1) = \deg(g), \tag{21}$$

and this finishes the proof. $\square$