1. The path-lifting lemma

We start with a basic technical lemma that will allow us to define the degree of a continuous map from the circle $S^1$ to itself.

Let $e : \mathbb{R} \to S^1$ be the “exponential map.” If we view $S^1$ as $\{ z \in \mathbb{C} \mid |z| = 1 \}$, then $e$ is given by

$$ e(t) = \exp(2\pi it). $$

Or, if we view $S^1$ as $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$, then $e$ is given by

$$ e(t) = (\cos(2\pi t), \sin(2\pi t)). $$

Clearly, $e$ is a continuous surjection. Since both the functions $t \mapsto \cos(2\pi t)$ and $t \mapsto \sin(2\pi t)$ are periodic of period $1$, we have

$$ e(t + C) = e(t), \quad \text{for every } C \in \mathbb{Z}. $$

Moreover,

$$ e(t_1) = e(t_2) \iff t_1 - t_2 \in \mathbb{Z}. $$

Finally, let $I = [0, 1]$ be the unit interval in $\mathbb{R}$, and denote by $e_0 : I \to S^1$ the restriction of $e$ to $I$. Again, $e_0$ is a continuous surjection—in fact, a quotient map.

**Lemma 1.1** (Path-Lifting Lemma). Let $g : [0, 1] \to S^1$ be a continuous map, and let $x \in \mathbb{R}$ such that $e(x) = g(0)$. There is then a unique continuous map $\tilde{g} : [0, 1] \to \mathbb{R}$ such that

- $e(\tilde{g}(t)) = g(t)$, for all $t \in [0, 1]$.
- $\tilde{g}(0) = x$.

A map $\tilde{g}$ as above is called a lift of $g$. Once we impose the “initial condition” $\tilde{g}(0) = x$, we get the unique lift of $g$ at $x$. The situation is summarized in the commuting diagrams

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\tilde{g}} & \mathbb{R} \\
\downarrow{e} & & \downarrow{e} \\
[0, 1] & \xrightarrow{g} & S^1 \\
& \searrow{\tilde{g}} & \\
& & 0 \xrightarrow{g} g(0)
\end{array}
$$

An analogous statement holds for homotopies.
Lemma 1.2 (Homotopy-Lifting Lemma). Let $F : I \times I \to S^1$ be a continuous map, and let $x \in \mathbb{R}$ such that $e(x) = F(0,0)$. There is then a unique continuous map $\tilde{F} : I \times I \to \mathbb{R}$ such that

- $e(\tilde{F}(t,s)) = g(t,s)$, for all $t,s \in I \times I$.
- $\tilde{F}(0,0) = x$.

2. The degree of a circle map

Let $f : S^1 \to S^1$ be a continuous map. Consider the composite

$$g = f \circ e_0 : [0,1] \to S^1.$$  

Clearly, $g$ is a continuous map. Moreover, since $e_0(0) = e_0(1) = (1,0)$, we have:

$$g(0) = g(1).$$

By Lemma 1.1, the map $g$ admits a lift $\tilde{g} : [0,1] \to \mathbb{R}$. That is to say, $e \circ \tilde{g} = g$. From (6), we get:

$$e(\tilde{g}(0)) = e(\tilde{g}(1))$$

Applying (4), we obtain:

$$\tilde{g}(1) - \tilde{g}(0) \in \mathbb{Z}$$

This leads to the following definition

Definition 2.1. With notation as above, the degree of the map $f : S^1 \to S^1$ is the integer $\deg(f)$ given by

$$\deg(f) = \tilde{g}(1) - \tilde{g}(0).$$

We must verify that $\deg(f)$ is well-defined, i.e., does not depend on the choice of lift $\tilde{g}$ for $g = f \circ e_0$. So, suppose $\bar{g} : [0,1] \to \mathbb{R}$ is another lift of $g$. Note that

$$e(\bar{g}(0)) = e(\bar{g}(0)) = g(0).$$

Thus, again by (4), we must have

$$\bar{g}(0) - \tilde{g}(0) = C,$$

for some $C \in \mathbb{Z}$.

Consider the map $\tilde{\bar{g}} : [0,1] \to \mathbb{R}$ given by

$$\tilde{\bar{g}}(t) = \tilde{g}(t) + C.$$  

We then have, by (3),

$$e(\tilde{\bar{g}}(t)) = e(\tilde{g}(t) + C) = e(\tilde{g}(t)) = g(t).$$

In other words, $\tilde{\bar{g}}$ is also a lift of $g$. Combining (9) and (11), we see that

$$\tilde{\bar{g}}(0) = \bar{g}(0) + C = \tilde{g}(0).$$
That is, the two lifts, $\tilde{g}$ and $\tilde{\tilde{g}}$, agree at 0. By the uniqueness statement from Lemma 1.1, these two lifts must agree, for all $t \in [0, 1]$; that is,

\[(14) \tilde{\tilde{g}} = \tilde{g}.\]

In view of (11), this is the same as saying

\[(15) \tilde{g}(t) = \tilde{\tilde{g}}(t) + C, \quad \text{for all } t \in [0, 1].\]

Therefore,

\[(16) \tilde{g}(1) - \tilde{g}(0) = (\tilde{g}(1) - C) - (\tilde{g}(0) - C) = \tilde{g}(1) - \tilde{\tilde{g}}(0),\]

showing that we obtain the same value for $\deg(f)$, whether we use the lift $\tilde{g}$, or the lift $\tilde{\tilde{g}}$ in Definition 2.1.

The following theorem shows that the degree of a circle map depends only on its homotopy class.

**Theorem 2.2.** Let $f, g: S^1 \to S^1$ be two continuous maps. Suppose $f \simeq g$. Then $\deg(f) = \deg(g)$.

**Proof.** Let $H: S^1 \times I \to S^1$ be a homotopy from $f$ to $g$. That is,

\[(17) H(z, 0) = f(z) \text{ and } H(z, 1) = g(z), \quad \text{for all } z \in S^1.\]

Consider the map $F: I \times I: S^1 \to S^1$ obtained by composing $H$ with the map $e_0 \times \text{id}_I$. By Lemma 1.2, the map $F$ lifts to a map $\tilde{F}: I \times I \to \mathbb{R}$; this maps fits into the commuting diagram

\[
\begin{array}{c}
\mathbb{R} \\
\downarrow \quad e \\
I \times I \xrightarrow{\tilde{g}} S^1 \times I \xrightarrow{H} S^1
\end{array}
\]

Clearly, the map $t \mapsto \tilde{F}(t, 0)$ is a lift of $f$, and likewise, the map $t \mapsto \tilde{F}(t, 1)$ is a lift of $g$. By the definition of degree, we have

\[(18) \deg(f) = \tilde{F}(1, 0) - \tilde{F}(0, 0)\]
\[(19) \deg(g) = \tilde{F}(1, 1) - \tilde{F}(0, 1)\]

Now consider the continuous map $G: [0, 1] \to \mathbb{Z}$ given by

\[(20) G(s) = \tilde{F}(1, s) - \tilde{F}(0, s), \quad \text{for all } s \in [0, 1].\]

Since the interval $[0, 1]$ is connected, and $\mathbb{Z}$ is discrete, the image of $G$ must be a singleton, i.e., $G$ is a constant function. Putting things together, we find:

\[(21) \deg(f) = G(0) = G(1) = \deg(g),\]

and this finishes the proof. \qed