Prof. Alexandru Suciu

MATH 4565 | TOPOLOGY | Spring 2010

MIDTERM EXAM

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1. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X.

(a) Is their union, $\mathcal{T} \cup \mathcal{T}'$, a topology on X? Why, or why not?

(b) Is their intersection, $\mathcal{T} \cap \mathcal{T}'$, a topology on X? Why, or why not?

2. Let $p\colon X \to Y$ be a continuous map. Suppose there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y. Show that p is a quotient map.

3. Let $f\colon X \to Y$ and $g\colon X \to Y$ be two continuous maps. Suppose Y is a Hausdorff space, and that there is a dense subset $D \subset X$ such that $f(x) = g(x)$ for all $x \in D$. Show that $f(x) = g(x)$ for all $x \in X$.

4. Let $X = \{1, 2, 3, 4\}$, equipped with the topology

$$\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}.$$

Let $Y = \{a, b, c\}$.

(a) Let $p\colon X \to Y$ be the function sending $1 \mapsto a$, $2 \mapsto a$, $3 \mapsto b$, $4 \mapsto c$. Find the quotient topology $\mathcal{T}_p$ on Y defined by the function p.

(b) Let $q\colon X \to Y$ be the function sending $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto b$, $4 \mapsto c$. Find the quotient topology $\mathcal{T}_q$ on Y defined by the function q.

(c) Are the spaces $(Y, \mathcal{T}_p)$ and $(Y, \mathcal{T}_q)$ homeomorphic? If yes, write down a specific homeomorphism. If not, explain why not.

5. Suppose X is homeomorphic to X' and Y is homeomorphic to Y'. Show that $X \times Y$ is homeomorphic to $X' \times Y'$. (Both products are equipped with the product topology.)

6. Show that $\mathbb{Z}$, equipped with the digital line topology, is not homeomorphic to $\mathbb{Z}$, equipped with the finite complement topology.

7. A space X is said to be *homogenous* if, for every two points $x_1, x_2 \in X$, there is a self-homeomorphism $f\colon X \to X$ such that $f(x_1) = x_2$. Prove that homogeneity is a topological property. That is to say, if X is homeomorphic to Y, and X is homogeneous, then Y is also homogeneous.

8. Let (X, d) be a metric space. Show that the distance function, $d\colon X \times X \to \mathbb{R}$, is continuous. (Here, X has the topology induced by d, and $X \times X$ has the product topology.)