Prof. Alexandru Suciu

MTH U371 **LINEAR ALGEBRA** **Spring 2006**

SOLUTIONS TO QUIZ 7

---

1. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 9 & -5 \end{bmatrix}$.

(a) Find the eigenvalues of A.

$$\begin{aligned} \det(A - \lambda I_3) &= (4-\lambda)\det\begin{bmatrix} 2-\lambda & 2 \\ 9 & -5-\lambda \end{bmatrix} \\ &= (4-\lambda)\big[(2-\lambda)(-5-\lambda) - 18\big] \\ &= (4-\lambda)(\lambda^2 + 3\lambda - 28) = -(\lambda-4)^2(\lambda+7) \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 4$ (with multiplicity 2), and $\lambda_2 = -7$.

(b) Find a basis for each eigenspace of A.

$$E_{\lambda_1} = \ker(A - \lambda_1 I_3) = \ker \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 9 & -9 \end{bmatrix} \text{ has basis the vectors } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$E_{\lambda_2} = \ker(A - \lambda_2 I_3) = \ker \begin{bmatrix} 11 & 0 & 0 \\ 0 & 9 & 2 \\ 0 & 9 & 2 \end{bmatrix} \text{ has basis the vector } \begin{bmatrix} 0 \\ -2 \\ 9 \end{bmatrix}$$

(c) Find a diagonal matrix D and an invertible matrix S such that $A = S \cdot D \cdot S^{-1}$.

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & 9 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -7 \end{bmatrix}.$$

2. A 4×4 matrix A has eigenvalues $\lambda_1 = -4$, $\lambda_2 = -1$, $\lambda_3 = 2$, $\lambda_4 = 3$.

(a) What is the characteristic polynomial of A?

$$\det(A - \lambda I_4) = (\lambda+4)(\lambda+1)(\lambda-2)(\lambda-3)$$

(b) Compute $\operatorname{tr}(A)$.

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$$

(c) Compute $\det(A)$.

$$\det(A) = \lambda_1\lambda_2\lambda_3\lambda_4 = 24$$

(d) What are the eigenvalues of A^2?

$$\lambda_1^2 = 16,\ \lambda_2^2 = 1,\ \lambda_3^2 = 4,\ \lambda_4^2 = 9$$

(e) Compute $\operatorname{tr}(A^2)$.

$$\operatorname{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 30$$

(f) Compute $\det(A^2)$.

$$\det(A) = \det(A)^2 = \lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 = 576$$

3. Let $D = \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix}$.

Note that D is a diagonal matrix, with distinct eigenvalues: $\lambda_1 = -3$ and $\lambda_2 = 7$.
Also, $\operatorname{tr}(D) = 4$ and $\det(D) = -21$.

(a) Let $A = \begin{bmatrix} 1 & 5 \\ 5 & 3 \end{bmatrix}$. Is A similar to D?

We have: $\operatorname{tr}(A) = 4$ and $\det(A) = -22$.
Thus A cannot be similar to D (the determinants are not equal).

(b) Let $B = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$. Is B similar to D?

We have: $\operatorname{tr}(B) = 4$ and $\det(B) = -21$.
Thus B is similar to D (the two matrices have the same trace and determinant, and thus the same eigenvalues; moreover, B is diagonalizable, since the eigenvalues are distinct; thus, its diagonalization must be D).

(c) Let $C = \begin{bmatrix} -4 & -3 \\ 5 & 9 \end{bmatrix}$. Is C similar to D?

We have: $\operatorname{tr}(C) = 5$ and $\det(C) = -21$.
Thus C cannot be similar to D (the traces are not equal).

4. A 2×2 matrix A matrix has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$, with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Find A.

$$A = S \cdot D \cdot S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

(b) Consider the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$, with initial value $\vec{x}(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.
Find a closed form for $\vec{x}(t)$.

$$\begin{aligned} \vec{x}(t) &= A^t \cdot \vec{x}(0) = S \cdot D^t \cdot S^{-1} \cdot \vec{x}(0) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^t & 0 \\ 0 & 5^t \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2^t & 5^t \\ 0 & 5^t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2^t + 3 \cdot 5^t \\ 3 \cdot 5^t \end{bmatrix} \end{aligned}$$

or:

$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 = 2^t \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot 5^t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$