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MTH U371 **LINEAR ALGEBRA** **Spring 2005**

Solutions to Quiz 4

1. [8 points] Apply the Gram-Schmidt process to the vectors $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, and write the result in the form $A = Q \cdot R$.

$$r_{11} = \|\vec{v}_1\| = \sqrt{9+1} = \sqrt{10}$$

$$\vec{u}_1 = \frac{1}{r_{11}}\vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \tfrac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix} = -\frac{6}{\sqrt{10}} = -\frac{3}{5}\sqrt{10}$$

$$\vec{w}_2 = \vec{v}_2 - r_{12}\vec{u}_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} - \left(-\tfrac{3}{5}\sqrt{10}\right)\tfrac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \tfrac{3}{5}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$r_{22} = \|\vec{w}_2\| = \tfrac{1}{5}\sqrt{1+9} = \frac{\sqrt{10}}{5}$$

$$\vec{u}_2 = \frac{1}{r_{22}}\vec{w}_2 = \tfrac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$A = Q \cdot R$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_1 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} = \left(\tfrac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\right) \cdot \left(\sqrt{10}\begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{bmatrix}\right)$$

2. 7 points

Consider the vectors $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 1 \end{bmatrix}$.

(a) Find the matrix of the orthogonal projection onto the line L in $\mathbb{R}^4$ spanned by $\vec{v}$.

$$\vec{u} = \frac{1}{\|\vec{v}\|}\,\vec{v} = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

$$A = \vec{u}\cdot\vec{u}^\top = \tfrac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \cdot \tfrac{1}{3}\begin{bmatrix} 1 & 0 & 2 & -2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & -4 \\ -2 & 0 & -4 & 4 \end{bmatrix}$$

(b) Find the projection of $\vec{w}$ onto the line L.

$$\mathrm{proj}_L(\vec{w}) = A\vec{w} = \frac{1}{9}\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & -4 \\ -2 & 0 & -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \frac{8}{9}\begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

3. 6 points

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a 3×3 matrix.

(a) Is the matrix $B = A^\top A A^\top$ symmetric? Justify your answer.

We have

$$B^\top = (A^\top A A^\top)^\top = (A^\top)^\top A^\top (A^\top)^\top = A A^\top A$$

and this does not equal B, in general. Thus B is not symmetric.

(b) Is the matrix $B = 2A + 2A^\top$ symmetric? Justify your answer.

We have

$$B^\top = (2A + 2A^\top)^\top = 2A^\top + 2(A^\top)^\top = 2A^\top + 2A = B$$

Thus B is symmetric.

(c) Suppose A is orthogonal. What is A^{-1}?

A orthogonal means $AA^\top = I_3$.
Hence A is invertible, with inverse

$$A^{-1} = A^\top = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

4. 9 points

(a) Find the least squares solution $\vec{x}^*$ of the inconsistent system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A^\top A\vec{x}^* = A^\top B$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 4 \\ 4 & 2 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\vec{x}^* = \begin{bmatrix} 11 & 4 \\ 4 & 2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & -4 \\ -4 & 11 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\vec{x}^* = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{6} \end{bmatrix}$$

(b) Use your answer to part (a) to find the projection of $\vec{b}$ onto $\operatorname{im} A$.

$$\operatorname{proj}_{\operatorname{im} A}(\vec{b}) = A\vec{x}^* = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 13 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{13}{6} \\ \frac{5}{6} \\ \frac{2}{3} \end{bmatrix}$$

(c) Determine the error $\|\vec{b} - A\vec{x}^*\|$.

$$\|\vec{b} - A\vec{x}^*\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 13 \\ 5 \\ 4 \end{bmatrix} \right\| = \left\| \frac{7}{6} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\| = \frac{7}{6}\sqrt{6} = \frac{7}{\sqrt{6}}$$