Prof. A. Suciu · MTH U371–LINEAR ALGEBRA · Spring 2005

PRACTICE FINAL EXAM

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1. Are the following vectors independent or dependent? If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ -4 \\ -7 \end{bmatrix}$$

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2. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Are the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ linearly independent or dependent? If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

(b) Write the vector $\vec{b} = \begin{bmatrix} 1 \\ -7 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

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3. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$.

(a) Are the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ linearly independent or dependent? If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

(b) Write the vector $\vec{b} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ as a linear combination of the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

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4. Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 5 & 7 \\ -6 & 10 & -7 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -2 \\ -3 \\ 8 \end{bmatrix}$.

(a) Solve the system of linear equations $A\vec{x} = \vec{b}$, indicating clearly the row operations, pivots, leading variables, and free variables.

(b) Find a basis for the image of A.

(c) Find a basis for the kernel of A.

(d) Find the rank of A.

(e) Find a non-zero vector that is orthogonal to both the vectors $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ 10 \\ -7 \end{bmatrix}$.

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5. The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 2 & 3 & 5 & 2 & 1 \\ 3 & 5 & 8 & 3 & 1 \\ 4 & 7 & 11 & 4 & 1 \end{bmatrix}$ has the matrix $E = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ as its row-reduced echelon form.

(a) Find a basis for the image of A.

(b) Find a basis for the kernel of A.

(c) Compute: $\operatorname{rank} A$, $\dim(\operatorname{im} A)$, $\dim(\ker A)$, $\dim(\operatorname{im} A^\top)$, $\dim(\ker A^\top)$.

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6. The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}$ has the matrix $E = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ as its row-reduced echelon form.

(a) Find a basis for the image of A.

(b) Find a basis for the kernel of A.

(c) Compute: $\operatorname{rank} A$, $\dim(\operatorname{im} A)$, $\dim(\ker A)$, $\dim(\operatorname{im} A^\top)$, $\dim(\ker A^\top)$.

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7. (a) Find the 3×3 matrix A associated with the linear mapping that rotates the yz-plane by $-60°$ and reflects the x-axis about the yz-plane.

(b) Is A orthogonal?

(c) What is $\det(A)$?

(d) What is the image of the point $(-3, 2, 1)$ under the above mapping?

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8. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ which is a counterclockwise rotation of $30°$ about the y-axis, followed by a dilation by a factor of 6.

(a) Find the matrix A corresponding to T.

(b) What is the image of the vector $\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ under the map T?

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9. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that rotates the xz-plane by $120°$ and reflects the y-axis about the xz-plane.

(a) Find the matrix A corresponding to T.

(b) What is $\det(A)$?

(c) Is A orthogonal? Why, or why not?

(d) Find A^{-1}.

(e) What is the image of the vector $\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ under the map T?

10. In each of the following, a vector space V and a subset S are given. Circle one answer:

(a) $V = \mathbb{R}^4$, $S = \{(-t, 4t, 3t, 0) \mid t \text{ is a real number}\}$

S is closed under addition:	YES	NO
S is closed under scalar multiplication:	YES	NO
S is a vector subspace of V:	YES	NO

(b) $V = \mathbb{R}^4$, $S = \{(-t, 4t, 3t, 1) \mid t \text{ is a positive real number}\}$

S is closed under addition:	YES	NO
S is closed under scalar multiplication:	YES	NO
S is a vector subspace of V:	YES	NO

(c) $V = \mathbb{R}^4$, $S = \{x \in V \mid x_1 + 2x_2 - x_3 = 0, \quad 2x_1 + x_3 - 3x_4 = 0\}$

S is closed under addition:	YES	NO
S is closed under scalar multiplication:	YES	NO
S is a vector subspace of V:	YES	NO

(d) $V = \mathbb{R}^4$, $S = \{x \in V \mid x_1 + 2x_2 - x_3 \geq 0\}$

S is closed under addition:	YES	NO
S is closed under scalar multiplication:	YES	NO
S is a vector subspace of V:	YES	NO

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11. In each of the following cases, determine whether or not the given subset V of $\mathbb{R}^n$ is a vector subspace. If it is, identify it as either the kernel or the image of a matrix A, and write down the matrix A. If it is not a vector subspace, explain why not.

(a) $V = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \text{ where } t \text{ and } s \text{ take all real values}\}$

(b) $V = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \text{ where } t \text{ takes all real values}\}$

(c) $V = \{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 x_4 = 0\}$

(d) $V = \{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 + 3x_3 = 0, \quad x_3 - x_4 = 0, \quad 2x_1 + x_3 + x_4 = 0\}$

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12. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$.

(a) Find the least squares solution $\vec{x}^*$ of the inconsistent system $A\vec{x} = \vec{b}$.

(b) Use your answer to part (a) to find the projection of $\vec{b}$ onto the image of A.

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13. Let $A = \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}$.

(a) Find the least squares solution $\vec{x}^*$ of the inconsistent system $A\vec{x} = \vec{b}$.

(b) Find the 4×4 matrix associated with the projection of $\mathbb{R}^4$ onto the subspace $\operatorname{im} A$.

(c) Find the projection of $\vec{b}$ onto $\operatorname{im} A$.

14. The number of students getting an A on the Spring final exam of a certain Linear Algebra course is as follows:

Year	1997	1998	1999	2000
A's	2	1	4	6

Represent the years $1997, 1998, 1999, 2000$ as $0, 1, 2, 3$, respectively, and let t denote the year (after 1997). Let y denote the number of A's.

(a) Find the line $y = mt + b$ that best fits the above data points, using the least squares method.

(b) Use the equation obtained in part (a) to estimate how many students will get an A in Linear Algebra in Spring 2001.

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15. Let $\mathbf{a}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

(a) Find unit vectors in the direction of $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively.

(b) Find the lengths of $\mathbf{a}_1$ and $\mathbf{a}_2$, and compute the dot product $\mathbf{a}_1 \cdot \mathbf{a}_2$.

(c) Find the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$. Are $\mathbf{a}_1$ and $\mathbf{a}_2$ orthogonal?

(d) Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. Use the Gram-Schmidt process to find the QR-factorization of A.

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16. Apply the Gram-Schmidt process to the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and write the result in the form $A = QR$.

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17. Consider the independent vectors $\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Use the Gram-Schmidt process to find an othonormal basis, $\vec{u}_1, \vec{u}_2, \vec{u}_3$, for the space spanned by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

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18. Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$.

(a) Find the characteristic equation for A.

(b) Find the eigenvalues of A.

(c) Find a basis for each eigenspace of A.

(d) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

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19. Let $A = \begin{bmatrix} 5 & 6 & 0 \\ 7 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A.

(c) Find a basis for each eigenspace of A.

(d) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

20. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 9 & -5 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A.

(c) Find a basis for each eigenspace of A.

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21. A 5×5 matrix A has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 2$, $\lambda_4 = 3$, $\lambda_5 = 4$.

(a) Compute tr A, det A.

(b) Compute tr A^2, det A^2.

(c) Compute $\det(3I_5 - A)$.

(d) Compute $\det(3A)$.

(e) Is A invertible? Why, or why not?

(f) Is A orthogonal? Why, or why not?

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22. A 4×4 matrix A has eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\lambda_4 = 4$.

(a) What is the characteristic polynomial of A?

(b) Compute $\operatorname{tr}(A)$ and $\det(A)$.

(c) Compute $\det(-2A)$.

(d) Compute $\det(A + 2I_4)$.

(e) What are the eigenvalues of A^3?

(f) Compute $\operatorname{tr}(A^3)$ and $\det(A^3)$.

(g) Is A invertible? Why, or why not?

(h) Is A orthogonal? Why, or why not?

(i) Is A diagonalizable? Why, or why not?

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23. Find a 2×2 matrix A such that $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of A, with eigenvalues -2 and 5, respectively.

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24. Let $A = \begin{bmatrix} 27 & -12 \\ 56 & -25 \end{bmatrix}$. Write A^t (the matrix A raised to the power t, a positive integer) in the form of a single 2×2 matrix. (You may use the fact that the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector with associated eigenvalue 3, and the vector $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ is an eigenvector with associated eigenvalue -1.)

25. A 2×2 matrix A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$, with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Write A^t (for t an integer) in the form of a single 2×2 matrix.

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26. Suppose a 3×3 matrix A has an eigenvalue $\lambda_1 = 4$ and two complex conjugate eigenvalues λ_2 and λ_3. Suppose also $\text{tr}(A) = 8$ and $\det(A) = 52$.

(a) Find λ_2 and λ_3.

(b) Find the eigenvalues of A^2.

(c) Compute $\text{tr}(A^2)$ and $\det(A^2)$.

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27. Solve the discrete dynamical system $\vec{x}(t+1) = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} \vec{x}(t)$ with initial condition $\vec{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Write down the formula for $\vec{x}(t)$, and sketch the phase portrait (indicate the past and the future).

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28. Solve the discrete dynamical system $\vec{x}(t+1) = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \vec{x}(t)$ with initial condition $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use your solution to compute $\vec{x}(1)$ and $\vec{x}(2)$. What is $\lim_{t\to\infty} \vec{x}(t)$?

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29. For the symmetric matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix}$, find an orthogonal matrix S and a diagonal matrix D such that $A = SDS^\top$.

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30. For the symmetric matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find an orthogonal matrix S and a diagonal matrix D such that $A = SDS^\top$.

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31. Find the singular value decomposition for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

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32. Find the singular value decomposition for the matrix $A = \begin{bmatrix} 6 & 3 \\ -1 & 2 \end{bmatrix}$.
