MTH U345 **Ordinary Differential Equations** **Fall 2008**

Lab 1: Partial solution

Exercise 2(d) *Consider the initial value problem*

$$dy/dt = y(y-2), \quad y(0) = 3.$$

What happens to $y(t)$ as t increases? Can you show that $y(t)$ escapes to infinity?

First, solve the ODE by separating variables, and integrating both sides using partial fractions:

$$\begin{aligned}
\frac{dy}{dt} &= y(y-2) \\
\int \frac{dy}{y(y-2)} &= \int dt + C \\
\frac{1}{2}\int \frac{dy}{y-2} - \frac{1}{2}\int \frac{dy}{y} &= t + C \\
\frac{1}{2}\ln\left|\frac{y-2}{y}\right| &= t + C \\
\left|\frac{y-2}{y}\right| &= e^{2t+2C} \\
\frac{y-2}{y} &= \pm K e^{2t} \qquad \text{with } K = e^{2C} > 0
\end{aligned}$$

Now use the initial condition $y(0) = 3$ to figure out both the constant K and the sign:

$$\begin{aligned}
\frac{3-2}{3} &= \pm K \cdot 1 \\
K &= \frac{1}{3} \qquad \text{and the sign is } +
\end{aligned}$$

Thus

$$\begin{aligned}
y - 2 &= \tfrac{1}{3} y e^{2t} \\
(1 - \tfrac{1}{3}e^{2t})y &= 2 \\
\boxed{y(t) = \frac{6}{3 - e^{2t}}}
\end{aligned}$$

Notice that $y(t)$ is defined as long as the denominator does not vanish. The denominator $3 - e^{2t}$ equals 0 precisely when $e^{2t} = 3$, i.e.,

$$\boxed{t = \frac{\ln 3}{2} \simeq 0.5493}$$

Notice that the solution curve $y(t) = \frac{6}{3-e^{2t}}$ is only defined for $t < \frac{\ln 3}{2}$. As t approaches this limiting value from the left, $y(t)$ escapes to infinity:

$$\boxed{\lim_{t \to \frac{\ln 3}{2}^-} = +\infty}$$