Prof. Alexandru Suciu

MATH 4565 **TOPOLOGY** **Spring 2016**

MIDTERM EXAM

---

1. A space X is said to be *homogeneous* if, for every two points $x_1, x_2 \in X$, there is a self-homeomorphism $f\colon X \to X$ such that $f(x_1) = x_2$. Prove that homogeneity is a topological property. That is to say, if X is homeomorphic to Y, and X is homogeneous, then Y is also homogeneous.

2. Let $(X, \mathcal{T})$ be a topological space. Show that the following conditions are equivalent:

(a) X is locally connected.

(b) The family of open connected subsets of X is a basis for $\mathcal{T}$.

3. Prove or disprove the following:

(a) If X and Y are path-connected, then $X \times Y$ is path-connected.

(b) If $A \subset X$ is path-connected, then $\overline{A}$ is path-connected.

(c) If X is locally path-connected, and $A \subset X$, then A is locally path-connected.

(d) If X is path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is path-connected.

(e) If X is locally path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is locally path-connected.

4. Let $\mathbb{Z}$ be the set of integers. An *arithmetic progression* is a subset of the form $A_{a,b} = \{a + nb \mid n \in \mathbb{Z}\}$, with $a, b \in \mathbb{Z}$ and $b \neq 0$.

(a) Prove that the collection of arithmetic progressions,

$$\mathcal{A} = \{A_{a,b} \mid a, b \in \mathbb{Z},\ b \neq 0\},$$

is a basis for a topology on $\mathbb{Z}$.

(b) Is $\mathbb{Z}$ endowed with this topology a Hausdorff space?

(c) Is $\mathbb{Z}$ endowed with this topology a compact space?

5. Let $f\colon X \to Y$ be a continuous map. We say that f is *proper* if $f^{-1}(K)$ is compact, for every compact subset $K \subset Y$. We also say that f is *perfect* if f is surjective, closed, and $f^{-1}(\{y\})$ is compact for every point $y \in Y$.

(a) Show that every continuous map from a compact space to a Hausdorff space is both proper and closed.

(b) Show that every homeomorphism is a perfect map. Conversely, show that every injective perfect map is a homeomorphism.

(c) Give an example of a perfect map which is not open.

6. Let $f\colon X \to Y$ be a continuous map from a space X to a Hausdorff space Y. Let C be a closed subspace of Y, and let U be an open neighborhood of $f^{-1}(C)$ in X.

(a) Show that if X is compact then there is an open neighborhood V of C in Y such that $f^{-1}(V)$ is contained in U.

(b) Give an example to show that if X is not compact, then there need not be such a neighborhood V.

7. Let A be a subspace of a topological space X. A *retraction* of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. If such a map exists, we say that A is a *retract* of X.

(a) Prove the following: If X is Hausdorff and A is a retract of X, then A is closed.

(b) By the above, the open interval $(0, 1)$ is *not* a retract of the real line $\mathbb{R}$. Nevertheless, show that the closed interval $[0, 1]$ *is* a retract of $\mathbb{R}$.

8. Let $f\colon X \to Y$ and $g\colon X \to Y$ be two continuous maps. Suppose Y is a Hausdorff space, and that there is a dense subset $D \subset X$ such that $f(x) = g(x)$ for all $x \in D$. Show that $f(x) = g(x)$ for all $x \in X$.