Prof. Alexandru Suciu

MATH 4565 **TOPOLOGY** **Fall 2022**

MIDTERM EXAM

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1. Let $\mathbb{Z}$ be the set of integers, and let $\mathcal{B}$ be the set of all subsets of $\mathbb{Z}$ of the form $\{n\}$ with n odd, or $\{n-1, n, n+1\}$ with n even.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

(b) Is $\mathbb{Z}$ equipped with this topology a Hausdorff space?

(c) Is $\mathbb{Z}$ equipped with this topology a connected space?

2. Let $A \subset \mathbb{R}$ be the set of integers, viewed as a subspace of the reals, and let X be the quotient space $\mathbb{R}/A$ obtained by collapsing A to a point. Show that $X \cong \bigvee_{\mathbb{Z}} S^1$; that is, X is homeomorphic to a wedge sum of countably infinitely many circles.

3. Let $q\colon X \to Y$ be a quotient map, let U be an open subset of X, and let $q|_U\colon U \to q(U)$ be the restriction of q to U (and co-restricted to its image).

(a) Suppose U is a saturated open subset of X. Show that $q|_U\colon U \to q(U)$ is again a quotient map.

(b) Give an example showing that the saturation hypothesis is necessary.

4. Let $q\colon X \to Y$ be a quotient map. Suppose Y is connected, and, for each $y \in Y$, the subspace $q^{-1}(\{y\})$ is connected. Show that X is also connected.

5. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$, with the topology induced from the standard (Euclidean) topology on $\mathbb{R}^{n+1}$, for $n \geq 1$. Also let $\mathbb{RP}^n = S^n/\sim$ be the projective n-space, obtained as the quotient space of S^n by the equivalence relation $x \sim y$ if $y = x$ or $y = -x$.

(a) Show that S^n is path-connected, by constructing for any two points $x, y \in S^n$ an explicit path connecting them.

(b) Show that S^n is locally path-connected.

(c) Show that $\mathbb{RP}^n$ is path-connected and locally path-connected.

6. Let X be a topological space, and let A be a subset of X which is both open and closed. Show that A is a union of connected components of X.

7. Let X be a locally path-connected space. Let U be an open, connected subset of X. Show that U is path-connected.

8. Prove or disprove the following:

(a) If X and Y are path-connected, then $X \times Y$ is path-connected.

(b) If $A \subset X$ is path-connected, then $\overline{A}$ is path-connected.

(c) If X is locally path-connected, and $A \subset X$, then A is locally path-connected.

(d) If X is path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is path-connected.

(e) If X is locally path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is locally path-connected.