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MATH 4565 **TOPOLOGY** **Fall 2022**

Homework 4

Definition 1. Let X and Y be two topological spaces. A map $f\colon X \to Y$ is *proper* if the preimage under f of any compact set is compact.

Definition 2. A topological space X is said to be *compactly generated* if the following condition is satisfied: A subspace A is closed in X if and only if $A \cap K$ is closed in K for all compact subspaces $K \subseteq X$.

Definition 3. A topological space X is *locally compact* if the following condition is satisfied: For every point $x \in X$, there is a compact subset $K \subseteq X$ that contains an (open) neighborhood of x.

1. Suppose X is compact, and Y is Hausdorff. Show that every continuous map $f\colon X \to Y$ is both closed and proper.

2. Let $X \times Y$ be the direct product of two topological spaces, and let $p\colon X \times Y \to X$ be the first-coordinate projection map. Show that p is proper if and only if Y is compact.

3. Show that every locally compact space is compactly generated.

4. Let $f\colon X \to Y$ be a proper, continuous map, and assume Y is a compactly generated, Hausdorff space. Show that f is a closed map.

5. Let $f\colon X \to Y$ be a continuous map from a compact space X to a Hausdorff space Y. Let C be a closed subspace of Y, and let U be an open neighborhood of $f^{-1}(C)$ in X. Show that there is an open neighborhood V of C in Y such that $f^{-1}(V)$ is contained in U.

6. Let (X, d) be a metric space, and let $f\colon X \to X$ be a continuous function which has no fixed points.

(a) If X is compact, show that there is a real number $\epsilon > 0$ such that $d(x, f(x)) > \epsilon$, for all $x \in X$.

(b) Show that the conclusion in (a) is false if X is not assumed to be compact.