Prof. Alexandru Suciu

MATH 4565 **TOPOLOGY** **Fall 2022**

FINAL EXAM

---

1. Let X be a topological space. Suppose X contains an infinite, closed, discrete subspace. Show that X is *not* compact.

2. Let X be a topological space and let CX be the cone on X.

(a) Show that CX is contractible.

(b) Show that CX is locally connected if and only if X is locally connected.

3. Let X be a topological space and let $f\colon X \to S^n$ a continuous map to the n-sphere. Prove the following: If f is not surjective, then f is homotopic to a constant map.

4. Let $p\colon E \to B$ be a covering map. Suppose E is path-connected, and $\pi_1(B, b_0) = 0$. Show that p is a homeomorphism.

5. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, and let $p\colon \mathbb{R} \to S^1$ be the standard covering map given by $p(t) = e^{2\pi i t}$. Consider the product covering map $p \times p\colon \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, and let $f\colon [0,1] \to S^1 \times S^1$ be the loop given by $f(t) = (e^{4\pi i t}, e^{6\pi i t})$. Find the lift $\tilde{f}\colon [0,1] \to \mathbb{R}^2$ of f at $(0,0)$, and sketch both f and $\tilde{f}$.

6. A subspace $A \subset X$ is called a *deformation retract* of X if there is a retraction $r\colon X \to A$ with the property that $i \circ r \simeq \mathrm{id}_X$. Prove the following:

(a) Let $B \subset A \subset X$. If A is a deformation retract of X and B is a deformation retract of A, then B is a deformation retract of X.

(b) If A is a retract of X and X is contractible, then A is also contractible, and A is a deformation retraction of X.

7. Let X be a topological space, let $A \subset X$ be a subspace, and let $i\colon A \hookrightarrow X$ the inclusion map. Fix a basepoint $a_0 \in A$, and consider the induced homomorphism on fundamental groups, $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$.

(a) Suppose A is a retract of X. Show that i_* is injective.

(b) Give an example of an inclusion $i\colon A \hookrightarrow X$ where i_* is *not* injective.

(c) Suppose A is a deformation-retract of X. Show that i_* is an isomorphism.

(d) Give an example of an inclusion $i\colon A \hookrightarrow X$ that admits a retraction $r\colon X \to A$ for which i_* is *not* an isomorphism.

8. Let A be a 3×3 matrix with strictly positive real entries. Let $\alpha\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\alpha(v) = A \cdot v$ be the linear map defined by (left) multiplication by A and let

$$S = \{v = (v_1, v_2, v_3) \in \mathbb{R}^3 : v_1^2 + v_2^2 + v_3^2 = 1,\ v_i \geq 0\}$$

be the region on the unit sphere S^2 cut out by the positive orthant in $\mathbb{R}^3$.

(a) Use the map α to construct a continuous map $f\colon S \to S$.

(b) Show that S is homeomorphic to a 2-disk D^2.

(c) Use parts (a) and (b) to show that A has a strictly positive real eigenvalue.