Prof. Alexandru Suciu

MATH 4565 **TOPOLOGY** **Fall 2021**

MIDTERM EXAM

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1. Let $\mathbb{Z}$ be the set of integers. An *arithmetic progression* is a subset of the form $A_{a,b} = \{a + nb \mid n \in \mathbb{Z}\}$, with a, b integers and $b \neq 0$.

(a) Show that the collection of arithmetic progressions, $\mathcal{A} = \{A_{a,b} \mid a, b \in \mathbb{Z}, b \neq 0\}$, is a basis for a topology on $\mathbb{Z}$.

(b) Is $\mathbb{Z}$ endowed with this topology a Hausdorff space?

(c) Is $\mathbb{Z}$ endowed with this topology a connected space?

2. Let A be a subspace of a topological space X. A *retraction* of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. If such a map exists, we say that A is a *retract* of X.

(a) Prove the following: If X is Hausdorff and A is a retract of X, then A is closed.

(b) By the above, the open interval $(0, 1)$ is *not* a retract of the real line $\mathbb{R}$. Nevertheless, show that the closed interval $[0, 1]$ *is* a retract of $\mathbb{R}$.

3. Let $f\colon X \to Y$ and $g\colon X \to Y$ be two continuous maps. Suppose Y is a Hausdorff space, and that there is a dense subset $D \subset X$ such that $f(x) = g(x)$ for all $x \in D$. Show that $f(x) = g(x)$ for all $x \in X$.

4. Let $A \subset \mathbb{R}$ be the set of integers, and let X be the quotient space $\mathbb{R}/A$ obtained by collapsing A to a point. Show that $X \cong \bigvee_{\mathbb{Z}} S^1$; that is, X is homeomorphic to a wedge sum of countably infinitely many circles.

5. (a) Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ be the set of all points in the plane with at least one rational coordinate. Show that X, with the subspace topology, is a path-connected space.

(b) Let U be an *open*, connected subset of $\mathbb{R}^2$. Show that U is path-connected.

6. Show that $\mathbb{R}^n$, for $n > 1$, is not homeomorphic to any open subset of $\mathbb{R}$.

7. Let $f\colon X \to X$ be a continuous map. An element $x \in X$ is called a *fixed point* of f is $f(x) = x$. In each of the following situations, determine whether f must have a fixed point (in which case explain why), or does not necessarily have a fixed point (in which case give an example of a fixed-point-free map $f\colon X \to X$).

(a) $X = [0, 1]$.

(b) $X = [0, 1)$.

(c) $X = (0, 1)$.

8. Prove or disprove the following:

(a) If X and Y are path-connected, then $X \times Y$ is path-connected.

(b) If $A \subset X$ is path-connected, then $\overline{A}$ is path-connected.

(c) If X is locally path-connected, and $A \subset X$, then A is locally path-connected.

(d) If X is path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is path-connected.

(e) If X is locally path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is locally path-connected.