Prof. Alexandru Suciu

MATH 4565 **TOPOLOGY** **Fall 2021**

FINAL EXAM

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1. Let X be an infinite set, equipped with the finite complement topology.

(a) Determine whether X is compact (always, sometimes, or never).

(b) Determine whether X is Hausdorff (always, sometimes, or never).

2. Let $f\colon X \to Y$ be a continuous map from a space X to a Hausdorff space Y. Let C be a closed subspace of Y, and let U be an open neighborhood of $f^{-1}(C)$ in X.

(a) Show that if X is compact then there exists an open neighborhood V of C in Y such that $f^{-1}(V)$ is contained in U.

(b) Give an example to show that if X is not compact, then there need not be such a neighborhood V.

3. Let $\mathbb{RP}^2$ be the real projective plane, defined as the quotient space of $\mathbb{R}^3 \setminus \{0\}$ by the equivalence relation $\vec{x} \sim \vec{y}$ if $\vec{x} = \lambda\vec{y}$, for some $\lambda \in \mathbb{R} \setminus \{0\}$. Show that:

(a) $\mathbb{RP}^2$ is homeomorphic to the quotient space of the sphere S^2, obtained by identifying antipodal points.

(b) $\mathbb{RP}^2$ is compact.

(c) $\mathbb{RP}^2$ is path-connected.

4. Let X be a topological space and let CX be the cone on X.

(a) Show that CX is contractible.

(b) Show that CX is locally connected if and only if X is locally connected.

5. Let X be a topological space and let $f\colon X \to S^n$ a continuous map to the n-sphere. Prove the following: If f is not surjective, then f is homotopic to a constant map.

6. Let f and g be two paths in $\mathbb{R}^2 \setminus \{(0,0)\}$.

(a) Show that f is homotopic to g.

(b) Suppose f and g both start at $(-1,0)$ and end at $(0,1)$. Is f always path-homotopic to g?

7. Let X be a topological space, and let $B \subset A \subset X$ be subspaces. Prove the following:

(a) If A is a deformation retract of X and B is a deformation retract of A, then B is a deformation retract of X.

(b) If A is a retract of X and X is contractible, then A is also contractible, and A is a deformation retraction of X.

8. Let X be a topological space, let $A \subset X$ be a subspace, and let $i\colon A \hookrightarrow X$ the inclusion map. Fix a basepoint $a_0 \in A$, and consider the induced homomorphism on fundamental groups, $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$.

(a) Suppose A is a retract of X. Show that i_* is injective.

(b) Give an example of an inclusion $i\colon A \hookrightarrow X$ where i_* is *not* injective.

(c) Suppose A is a deformation-retract of X. Show that i_* is an isomorphism.

(d) Give an example of an inclusion $i\colon A \hookrightarrow X$ that admits a retraction $r\colon X \to A$ for which i_* is *not* an isomorphism.