Midterm Exam

MATH 3175 – Group Theory

Solutions

Problem 1. Recall that if $f : \mathbb{Z}_{15} \to D_6$ is a homomorphism then $|f(a)|$ divides $|a|$. Also recall, that mapping a generator of $\mathbb{Z}_{15}$ via f completely defines the homomorphism. The element 1 generates $\mathbb{Z}_{15}$, and $|1| = 15$, hence we have $|f(1)|$ must divide 15. I.e. $|f(1)|$ can be either 1, 3, 5, or 15. Since there are no elements of order 15 or 5 in D_6, we are left with the possibilities 1 and 3. If $|f(1)| = 1$ then we obtain the trivial homomorphism. Next, there are two elements of order 3 in D_6; these are r^2 and r^4. Hence, defining $f(1) = r^2$ gives rise to a homomorphism and similarly, defining $f(1) = r^4$ gives rise to another.

Problem 2. S_4 is the set of permutations on four elements. If we consider all the elements of S_4 which keep element 1 fixed, we obtain all permutations of elements 2,3 and 4 (with 1 being sent to itself). This subset is isomorphic to S_3 under the function which re-labels element 2 to element 1, element 3 to element 2 and element 4 to element 3. For example, the permutation $(1)(243) \mapsto (123)$. 3 More such permutation groups can be obtained by fixing element 2, 3 or 4 as we did above and are all isomorphic to S_3 via similar maps. If you did the computations out, you noticed that these groups were generated by $\langle(234), (23)\rangle$, $\langle(134), (13)\rangle$, $\langle(124), (12)\rangle$, and $\langle(123), (12)\rangle$ respectively.

Problem 3 a). Consider $f\colon \mathbb{R} \to T$ defined by $f(x) = e^{ix} = \sin(x)+i\cos(x)\ \forall\, x \in \mathbb{R}$. Note that the equality utilizes Euler's formula. First, notice that $f(ab) = f(a + b) = e^{i(a+b)} = e^{ia}e^{ib} = f(a)f(b)$, so f is a group homomorphism. Next, we will show surjectivity. Let $t \in T$. Then $t = a + bi$ for some $a, b \in \mathbb{R}$ such that $\sqrt{a^2 + b^2} = 1 \iff a^2 + b^2 = 1$. We wish to find an $x \in \mathbb{R}$ such that $f(x) = a + bi$. I.e. we need $\sin(x) = a$ and $\cos(x) = b$. But, since $a^2 + b^2 = 1$, the point (a, b) lies on the unit circle, implying that there is an angle x such that $a = \sin(x)$ and $b = \cos(x)$. Thus, f is surjective.

b). We will use the first isomorphism theorem to show that T is isomorphic to $\mathbb{R}/\mathbb{Z}$. First, notice that $\text{Ker}f = \{n2\pi | n \in \mathbb{Z}\} = (2\pi)\mathbb{Z}$ (since $\mathbb{Z}$ under addition is a group). Thus, by the first isomorphism theorem, $\mathbb{R}/(2\pi)\mathbb{Z}$ is isomorphic to T. Also note that we can make a slight change to our function in part a: $f(x) = e^{2\pi ix} = \sin(2\pi x) + i\cos(2\pi x)$ and we can follow the same reasoning as in part a) to conclude that this is in fact a surjective homomorphism. Furthermore, now the kernel of this new homomorphism is the set of integers. Thus, similarly to the work above, we will get that $\mathbb{R}/\mathbb{Z}$ is isomorphic to T.

Problem 4. a) The Heisenberg group G of 3×3 upper-diagonal matrices with entries in $\mathbb{Z}_2$ and 1's down the diagonal is a subgroup of $\text{GL}_2(\mathbb{Z}_2)$. As a set, $G = \mathbb{Z}_2^3$ (thus, G has order 8), with group operation given by $(a, b, c)\cdot(a', b', c') = (a+a', b+b', c+c'+ab')$. Clearly, $(1, 0, 0)\cdot(0, 1, 0) = (1, 1, 1)$ is different from $(0, 1, 0) \cdot (1, 0, 0) = (1, 1, 0)$, and so G is not Abelian. The only elements in G that commute with all of G are of the form $(0, 0, c)$; thus, $Z(G) \cong \mathbb{Z}_2$.

b) By Lagrange's theorem, all subgroups of G must have order either 1, 2, 4, or 8. To start with, G has the following subgroups: the trivial subgroup $\{(0, 0, 0)\}$, the center $Z(G) = \langle(0, 0, 1)\rangle$, and G itself. By general principles (or by inspection), all these subgroups are normal.

Additionally, G has 4 subgroups of order 2, generated by the elements $(1, 0, 0)$, $(0, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$, respectively, and 3 subgroups of order 4; the first one is cyclic, generated by $(1, 1, 1)$, while the last two are the Klein 4-groups $\langle(1,0,0),(0,0,1)\rangle$ and $\langle(0,1,0),(0,0,1)\rangle$. All four subgroups of order 2 are non-normal (as can be seen by conjugating each one by a suitable element), whereas all three subgroups of order 4 are normal (since they have index 2).

c) The group G is not isomorphic to Q_8, since the orders of their elements don't match. For instance, G has five elements of order 2 (the generators of the 5 subgroups of order 2 identified above), whereas $Q_8 = \{\pm 1, \pm i, \pm j \pm k\}$ has only one, namely, -1.

On the other hand, G is isomorphic to D_4. Indeed, we know from the classification of groups of order 8 that there are three Abelian ones ($\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) and two non-Abelian ones (Q_8 and D_4); since we've seen that G is non-Abelian and is not Q_8, it must be D_4. Alternatively, we can define an explicit isomorphism from $G = \langle(1,1,1),(1,0,0)\rangle$ to $D_4 = \langle a, b \mid a^4 = b^2 = 1, ba = a^{-1}b\rangle$ by sending $(1,1,1)$ to a and $(1,0,0)$ to b; clearly, the relations among the two sets of generators match under this correspondence.

Problem 5. Let $\mathrm{C}_G(S) = \{g \in G \mid gsg^{-1} = s \text{ for all } s \in S\}$ and $\mathrm{N}_G(S) = \{g \in G \mid gSg^{-1} = S\}$ be the centralizer and the normalizer of a subset $S \subset G$.

a) Let $g, h \in \mathrm{C}_G(S)$. Then $(gh^{-1})s(gh^{-1})^{-1} = gh^{-1}shg^{-1} = gsg^{-1} = s$ for all $s \in S$, and so $gh^{-1} \in \mathrm{C}_G(S)$. Thus, $\mathrm{C}_G(S)$ is a subgroup of G.

Now let $g, h \in \mathrm{N}_G(S)$. Then $(gh^{-1})S(gh^{-1})^{-1} = gh^{-1}Shg^{-1} = gSg^{-1} = S$, and so $gh^{-1} \in \mathrm{N}_G(S)$. Thus, $\mathrm{N}_G(S)$ is a subgroup of G.

b) Clearly, $\mathrm{C}_G(S)$ is a subset (and thus a subgroup) of $\mathrm{N}_G(S)$. To show it's actually a *normal* subgroup, let $h \in \mathrm{C}_G(S)$ and $g \in \mathrm{N}_G(S)$. Then, for every $s \in S$, we have $(ghg^{-1})s(ghg^{-1})^{-1} = ghg^{-1}sgh^{-1}g^{-1}$. But $g^{-1}sg = s'$ for some $s' \in S$, and so $ghg^{-1}sgh^{-1}g = ghs'h^{-1}g^{-1} = gs'g^{-1} = s$. Thus, $ghg^{-1} \in \mathrm{C}_G(S)$, showing that $\mathrm{C}_G(S) \trianglelefteq \mathrm{N}_G(S)$.

c) Take $S = G$; then $\mathrm{C}_G(S) = Z(G)$, the center of G. Assume now that G is non-Abelian (e.g., $G = S_3$); then $Z(G)$ is properly contained in G, showing that $S \not\subseteq \mathrm{C}_G(S)$.

d) If $H \leq G$ then $hkh^{-1} \in H$ for all $h, k \in H$, and hence $h \in \mathrm{N}_G(H)$ for all $h \in H$. Therefore H is a subset (and thus a subgroup) of $\mathrm{N}_G(H)$.