Final Exam

MATH 3175–Group Theory

Due Tuesday August 18 at 10 am

Problem 1. Let $f\colon G \to H$ be a function between two groups, and let

$$K := \{(x, y) \in G \times H \mid f(x) = y\}$$

be its graph. Show that f is a homomorphism if and only if K is a subgroup of the direct product $G \times H$.

Problem 2. A few results you may need for this problem:

1. If m and n are relatively prime natural numbers, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.
2. If every Sylow subgroup of a group G is normal, then G is isomorphic to the product of its Sylow subgroups.

Now, onto the question.

(a) Show that every group of order 15 is isomorphic to $\mathbb{Z}_{15}$.

(b) Let G be a group of order 255. Show that G has a normal subgroup H of order 17.

(c) Show that G/H is cyclic. (Hint: What is the order of G/H?)

(d) Show that G has a normal subgroup K of order either 3 or 5. (Hint: Go back to assignment 5, problem 3.)

(e) Show that, in either case, G/K is Abelian.

(f) BONUS: Show that G is Abelian.

Problem 3. Let A_4 be the group of even permutations of the set $\{1, 2, 3, 4\}$. Consider the subgroups $H = \langle(123)\rangle$ and $K = \langle(12)(34), (13)(24)\rangle$.

(a) Write down all the **left** and **right** cosets of H in A_4. Be sure to indicate the elements of each coset.

(b) What is the order of H? What is the index of H in A_4? Is H a normal subgroup of A_4?

(c) Write down all the **left** and **right** cosets of K in A_4. Be sure to indicate the elements of each coset.

(d) What is the order of K? What is the index of K in A_4? Is K a normal subgroup of A_4?

(e) Find the intersection $H \cap K$. Is this a subgroup of A_4? Is this a normal subgroup of A_4?

(f) Find the (internal) product HK. Is this a subgroup of A_4? Is this a normal subgroup of A_4?

(g) Find the direct product $H \times K$ and identify it (up to isomorphism) as another well-known group of the same order.

Problem 4. Show that Q_8 (quaternion group of order 8) is a subgroup of the symmetric group S_8, but not a subgroup of the symmetric group S_5.

Problem 5. Prove that the groups of the following orders are **not** simple:

(a) Order 36.

(b) Order 56.

Problem 6. Classify up to isomorphism all the finite groups G which satisfy the following two conditions:

1. G is a factor group of $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.
2. The order of every element of G divides 24.

Hint: What is the minimum number of generators of $\mathbb{Z}^2$? What is the minimum number of generators of a quotient group of $\mathbb{Z}^2$? The answer should be in the form of a set $S = \{G_1, G_2, \ldots, G_n\}$ where each group G_i satisfies those two conditions, $G_i \ncong G_j$ for $i \neq j$, and if G is any group which satisfies the two conditions then $G \cong G_i$ for some $G_i \in S$.

Problem 7. Consider the group $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{Z}_5, a \neq 0 \right\}$.

(a) Find all the Sylow subgroups of G and determine which are normal and which are not normal.

(b) Is G a direct product of all its Sylow subgroups? (Hint: use your answer from part (a).)

Problem 8. Let G be a group and $A, B \subseteq G$ be arbitrary subsets. Then we define

$$[A, B] := \langle [a, b] \mid a \in A, b \in B \rangle$$

to be the subgroup generated by commutators between A and B. In this notation, the derived (commutator) subgroup may be expressed as $G' = [G, G]$. Similarly, $A' = [A, A]$ refers to the set of commutators of an arbitrary subset $A \subseteq G$.

(1) Show that $H \trianglelefteq G$ if and only if $[G, H] \leq H$.

(2) Prove that $K' \trianglelefteq G$ whenever $K \trianglelefteq G$.

(3) If $H, K \trianglelefteq G$ such that G/H and G/K are Abelian, show that $G/(H \cap K)$ is also Abelian.

(4) BONUS: If $|G'| = m$ then each element $x \in G$ has at most m conjugates.

Problem 9. We call a group homomorphism $\varphi\colon G \to H$ a *monomorphism* if $\varphi \circ f = \varphi \circ g \implies f = g$ for homomorphisms f, g. Dually, we call a homomorphism $\psi : G' \to H'$ an *epimorphism* if $f' \circ \psi = g' \circ \psi \implies f' = g'$ for any homomorphisms f', g'. Show that monomorphisms are injective and that epimorphisms are surjective.

(Hint: Try to show the latter by contradiction. In particular, assume that $\psi\colon G' \to H'$ is not surjective and try to define homomorphisms $f, g\colon H' \to \mathrm{Sym}(H'/\psi(G'))$ that contradict the supposition that ψ is an epimorphism.)