1. List all the elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_8$, and compute their orders.

<table>
<thead>
<tr>
<th>Element $(a,b)$</th>
<th>(0.0)</th>
<th>(1.0)</th>
<th>(0.1)</th>
<th>(1.1)</th>
<th>(0.2)</th>
<th>(1.2)</th>
<th>(0.3)</th>
<th>(1.3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order $</td>
<td></td>
<td>(a,b)</td>
<td></td>
<td>$</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Element $(a,b)$</th>
<th>(0.4)</th>
<th>(1.4)</th>
<th>(0.5)</th>
<th>(1.5)</th>
<th>(0.6)</th>
<th>(1.6)</th>
<th>(0.7)</th>
<th>(1.7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order $</td>
<td></td>
<td>(a,b)</td>
<td></td>
<td>$</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

2. Show that the group $U(9)$ is isomorphic to the direct product $\mathbb{Z}_2 \oplus \mathbb{Z}_3$, by describing explicitly an isomorphism $\phi : U(9) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

$$U(9) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_3$$

1 $\mapsto (0, 0)$
2 $\mapsto (1, 1)$
4 $\mapsto (0, 2)$
5 $\mapsto (1, 2)$
7 $\mapsto (0, 1)$
8 $\mapsto (1, 0)$

3. Consider the group $G = S_3 \oplus \mathbb{Z}_6$.

(a) Determine the set of orders of elements in $G$, that is, the set $\{|g| \mid g \in G\}$.

Orders of elements in $S_3$: 1, 2, 3;
Orders of elements in $\mathbb{Z}_6$: 1, 2, 3, 6;
Orders of elements in $S_3 \oplus \mathbb{Z}_6$: 1, 2, 3, 6.

(b) Prove that $G$ is not cyclic.

The order of $G$ is 36, but there are no elements of order 36 in $G$. Hence $G$ is not cyclic.
4. How many elements of order 7 are there in $\mathbb{Z}_{70} \oplus \mathbb{Z}_{490}$?

\[
\#\{\text{elements of order 7 in } \mathbb{Z}_{70}\} = \phi(7) = 6
\]
\[
\#\{\text{elements of order 7 in } \mathbb{Z}_{490}\} = \phi(7) = 6
\]
\[
\#\{\text{of elements of order 7 in } \mathbb{Z}_{70} \oplus \mathbb{Z}_{490}\} = \phi(7) \times 1 + \phi(7) \times \phi(7) + 1 \times \phi(7)
\]
\[
= 6 + 6 \times 6 + 6 = 48
\]

5. List all abelian groups (up to isomorphism) of order 72. Write each such group as a direct product of cyclic groups of prime power order.

\[
\begin{align*}
\mathbb{Z}_2^3 \oplus \mathbb{Z}_3^2 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_3^2 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \\
\mathbb{Z}_2^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3
\end{align*}
\]

**WARNING:** Since the problem asks you to write down the groups as direct products of cyclic groups of prime power order, you cannot write for example $\mathbb{Z}_{72}$ instead of $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$, even though the two groups are isomorphic (because $2^3$ is prime to $3^2$).

6. Let $G$ be an abelian group of order 108. Suppose that $G$ has exactly eight elements of order 3, and one element of order 2. Determine the isomorphism class of $G$.

The order of $G$ has prime factorization $108 = 2^2 \times 3^3$.
The abelian groups of order 108 (up to isomorphism) are:

\[
\begin{align*}
\mathbb{Z}_2^2 \oplus \mathbb{Z}_3^3 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_3 \text{ (This group satisfies the conditions.)} \\
\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \text{ (This group has 26 elements of order 3 and 1 element of order 2.)} \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^3 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_3 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3
\end{align*}
\]