1. (5 points) Let $\mathbb{R}$ be the additive group of real numbers, and let $\mathbb{R}^+$ be the multiplicative group of positive real numbers. Consider the map $\phi: \mathbb{R} \rightarrow \mathbb{R}^+$ given by $\phi(x) = 2^x$.

(a) Show that $\phi$ is an isomorphism from $\mathbb{R}$ to $\mathbb{R}^+$.

We need to show that $\phi$ is a bijection, and a homomorphism.

- $\phi$ injective. Suppose $2^x = 2^y$. Taking $\log_2$ on both sides, we get $x = y$.
- $\phi$ surjective. Let $y \in \mathbb{R}^+$. Then $y = \phi(x)$, where $x = \log_2 y$.
- $\phi$ a homomorphism. Compute: $\phi(x + y) = 2^{x+y} = 2^x 2^y = \phi(x)\phi(y)$.

(b) What is the inverse isomorphism?

$\phi^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R}$, given by $\phi^{-1}(y) = \log_2 y$

2. (4 points) Show that the automorphism group $\text{Aut}(\mathbb{Z}_{10})$ is isomorphic to a cyclic group $\mathbb{Z}_n$. What is $n$?

$\text{Aut}(\mathbb{Z}_{10}) \cong U(10) \cong \mathbb{Z}_4$

3. (6 points) Show that the following pairs of groups are not isomorphic. In each case, explain why.

(a) $U(12)$ and $\mathbb{Z}_4$.

$U(12)$ is not cyclic, since $|U(12)| = 4$, but $U(12)$ has no element of order 4. On the other hand, $\mathbb{Z}_4$ is cyclic. Thus, $U(12) \not\cong \mathbb{Z}_4$.

(b) $S_3$ and $\mathbb{Z}_6$.

$S_3$ is not abelian, since, for instance, $(12) \cdot (13) \neq (13) \cdot (12)$. On the other hand, $\mathbb{Z}_6$ is abelian (all cyclic groups are abelian.) Thus, $S_3 \not\cong \mathbb{Z}_6$.

(c) $S_4$ and $D_{12}$.

Each permutation of $S_4$ can be written as composition of disjoint cycles. So the only possible orders for the elements in $S_4$ are 1, 2, 3, and 4. On the other hand, there is an element of order 12 in $D_{12}$, for instance, the counter-clockwise rotation by 30 degrees. Thus, $S_4 \not\cong D_{12}$.
4. (5 points) Let G be a group, and let a be an element of order 30. How many left cosets of \( \langle a^6 \rangle \) in \( \langle a \rangle \) are there? List all these cosets. (Make sure to indicate all the elements in each coset.)

Solution.

\(|\langle a \rangle| = |a| = 30, |\langle a^6 \rangle| = |a^6| = 30 / \gcd(6, 30) = 5.\)

The number of cosets is \( |\langle a \rangle| / |\langle a^6 \rangle| = 30 / 5 = 6.\) They are

\[
\begin{align*}
\langle a \rangle &= \{ e, a^6, a^{12}, a^{18}, a^{24} \} \\
a\langle a \rangle &= \{ a, a^7, a^{13}, a^{19}, a^{25} \} \\
a^2\langle a \rangle &= \{ \cdots \} \\
a^3\langle a \rangle &= \{ \cdots \} \\
a^4\langle a \rangle &= \{ \cdots \} \\
a^5\langle a \rangle &= \{ \cdots \}
\end{align*}
\]

5. (5 points) Let \( S_3 \) be the group of permutations of the set \( \{1, 2, 3\} \). Consider the subgroup \( H = \{(1), (13)\} \).

(a) Write down all the left cosets of \( H \) in \( S_3 \). (Make sure to indicate all the elements in each coset.)

\[
H = \{(1), (1 3)\} \\
(1 2)H = \{(1 2), (1 3 2)\} \\
(1 2 3)H = \{(1 2 3), (2 3)\}
\]

(b) What is the index of \( H \) in \( S_3 \)?

\[
\]
6. (5 points) Suppose $G$ is a group of order 35.
   (a) What are the possible orders for the elements of $G$?

      According to Lagrange’s theorem, the order of an element in $G$ divides the order of the group. So the possible orders of elements are 1, 5, 7 and 35.

   (b) Suppose $G$ has an element of order 35. What is $G$?

      $G$ must be cyclic, and so $G = \mathbb{Z}_{35}$.

   (c) Suppose $G$ has precisely one subgroup of order 5, and one subgroup of order 7. What is $G$?

      Again, $G = \mathbb{Z}_{35}$.

      Indeed, suppose the subgroup of order 5 is $H$, and the one of order 7 is $K$. Then $H \cup K$ has

      $$1 + (5 - 1) + (7 - 1) = 11$$

      elements. Choose an element $a$ from $G \setminus (H \cup K)$. The order of $a$ is 1, 5, 7 or 35. We claim the order of $a$ is 35. The order of $a$ is not 1 because it is not the identity. The order of $a$ is neither 5 nor 7, for otherwise $a$ would generate a subgroup $\langle a \rangle$ of order 5 or 7, distinct from $H$ or $K$, and we know there is precisely one subgroup of order 5 (namely, $H$), and precisely one subgroup of order 7 (namely, $K$). Thus $G = \langle a \rangle$, and so $G$ is cyclic.