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MATH 3175 **Group Theory** **Fall 2010**

Solutions for Quiz 3

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1. (a) Draw the subgroup lattice of $\mathbb{Z}_{30}$.

$$\begin{tikzcd}
 & \langle 2\rangle \arrow[r, dash] \arrow[dr, dash] & \langle 6\rangle \arrow[dl, dash] \arrow[dr, dash] & \\
\langle 1\rangle \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \langle 3\rangle \arrow[dr, dash] & \langle 10\rangle \arrow[r, dash] \arrow[dl, dash] & \langle 0\rangle \\
 & \langle 5\rangle \arrow[r, dash] & \langle 15\rangle \arrow[ur, dash] &
\end{tikzcd}$$

(b) Make a table with all the elements of $\mathbb{Z}_{30}$, grouped according to their orders; how many elements of each possible order are there?

Order	Elements	# of elements
30	$1, 7, 11, 13, 17, 19, 23, 29$	8
15	$2, 4, 8, 14, 16, 22, 26, 28$	8
10	$3, 9, 21, 27$	4
6	$5, 25$	2
5	$6, 12, 18, 24$	4
3	$10, 20$	2
2	15	1
1	0	1

2. Let a be an element of a group G, and suppose a has order 24.

(a) List all the elements in the subgroup $\langle a^4\rangle$, together with their respective orders.

Elements	e	a^4	a^8	a^{12}	a^{16}	a^{20}
Orders	1	6	3	2	3	6

(b) What are the generators of the subgroup $\langle a^4\rangle$?

The subgroup $\langle a^4\rangle$ has two generators, a^4 and a^{20}, because these are the only elements in $\langle a\rangle$ of order 6, which is the order of the subgroup.

3. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 1 & 4 & 2 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{bmatrix}$, viewed as elements in S_6.

(a) Compute the product of α and β:

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 2 & 4 \end{bmatrix}$$

(b) Compute the inverse of α:

$$\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 5 & 1 & 3 \end{bmatrix}$$

(c) Compute the conjugate of β by α:

$$\alpha\beta\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 5 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{bmatrix}$$

(d) Do α and β commute?

Compute: $\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 2 & 4 \end{bmatrix}$.
So yes, α and β commute.

Alternative (and shorter) explanation:
from (c), we have $\alpha\beta\alpha^{-1} = \beta$, hence $\alpha\beta = \beta\alpha$.

4. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 9 & 4 & 7 & 1 & 2 & 8 & 5 & 10 & 6 \end{bmatrix}$, viewed as an element in S_{10}.

(a) Write α as products of disjoint cycles.

$\alpha = (1\ 3\ 4\ 7\ 8\ 5)(2\ 9\ 10\ 6)$.

(b) Find the order of α.

The order is the least common multiple of the orders of the two cycles. So it's $\operatorname{lcm}(6, 4) = 12$.

(c) Write α as a product of transpositions.

$\alpha = (1\ 5)(1\ 8)(1\ 7)(1\ 4)(1\ 3)(2\ 6)(2\ 10)(2\ 9)$.

(d) Find the parity of α.

By part (c) of the problem, α can be written as a product of 8 transpositions, hence its parity is even.

5. (a) How many permutations of order 5 are there in S_5?

Solution.
$$\frac{5!}{5} = 4 \times 3 \times 2 \times 1 = 24.$$

(b) How many permutations of order 5 are there in S_6?

Solution. If a permutation of six elements has order five, it should fix one element and permute the other five. So there are
$$\binom{6}{5} \cdot \frac{5!}{5} = 6 \times (4 \times 3 \times 2 \times 1) = 6 \times 24 = 144$$
permutations in S_6 which have order five. The expression in the parentheses is from Part (a) of the problem.

6. Find permutations α and β such that:

(a) $|\alpha| = 2$, $|\beta| = 2$, and $|\alpha\beta| = 2$.

Sample solution. Let $\alpha = (1\ 2)$, $\beta = (3\ 4)$. Then $\alpha\beta = (1\ 2)(3\ 4)$. About the order of $\alpha\beta$, since it's the product of cycles on two disjoint sets, $|\alpha\beta| = \mathrm{lcm}(|\alpha|, |\beta|) = 2$.

(b) $|\alpha| = 2$, $|\beta| = 2$, and $|\alpha\beta| = 3$.

Sample solution. Let $\alpha = (1\ 2)$, $\beta = (2\ 3)$. Then $\alpha\beta = (1\ 2\ 3)$.

(c) $|\alpha| = 2$, $|\beta| = 4$, and $|\alpha\beta| = 4$.

Sample solution. Let $\alpha = (1\ 2)$, $\beta = (3\ 4\ 5\ 6)$. Then $\alpha\beta = (1\ 2)(3\ 4\ 5\ 6)$.