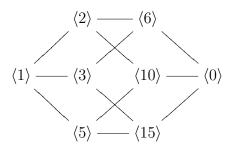
MATH 3175

Prof. Alexandru Suciu Group Theory

Solutions for Quiz 3

1. (a) Draw the subgroup lattice of \mathbb{Z}_{30} .



(b) Make a table with all the elements of \mathbb{Z}_{30} , grouped according to their orders; how many elements of each possible order are there?

Order	Elements	# of elements
30	1, 7, 11, 13, 17, 19, 23, 29	8
15	2, 4, 8, 14, 16, 22, 26, 28	8
10	3, 9, 21, 27	4
6	5, 25	2
5	6, 12, 18, 24	4
3	10, 20	2
2	15	1
1	0	1

- **2.** Let a be an element of a group G, and suppose a has order 24.
 - (a) List all the elements in the subgroup $\langle a^4 \rangle$, together with their respective orders.

Elements	e	a^4	a^8	a^{12}	a^{16}	a^{20}
Orders	1	6	3	2	3	6

(b) What are the generators of the subgroup $\langle a^4 \rangle$?

The subgroup $\langle a^4 \rangle$ has two generators, a^4 and a^{20} , because these are the only elements in $\langle a \rangle$ of order 6, which is the order of the subgroup.

3. Let
$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 1 & 4 & 2 \end{bmatrix}$$
 and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{bmatrix}$, viewed as elements in S_6 .

(a) Compute the product of α and β :

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 2 & 4 \end{bmatrix}$$

(b) Compute the inverse of α :

$$\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 5 & 1 & 3 \end{bmatrix}$$

(c) Compute the conjugate of β by α :

$$\alpha\beta\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 5 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{bmatrix}$$

(d) Do α and β commute?

Compute: $\beta \alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 2 & 4 \end{bmatrix}$. So yes, α and β commute.

Alternative (and shorter) explanation: from (c), we have $\alpha\beta\alpha^{-1} = \beta$, hence $\alpha\beta = \beta\alpha$.

- **4.** Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 9 & 4 & 7 & 1 & 2 & 8 & 5 & 10 & 6 \end{bmatrix}$, viewed as an element in S_{10} .
 - (a) Write α as products of disjoint cycles.

 $\alpha = (1 \ 3 \ 4 \ 7 \ 8 \ 5)(2 \ 9 \ 10 \ 6).$

(b) Find the order of α .

The order is the least common multiple of the orders of the two cycles. So it's lcm(6, 4) = 12.

(c) Write α as a product of transpositions.

 $\alpha = (1\ 5)(1\ 8)(1\ 7)(1\ 4)(1\ 3)(2\ 6)(2\ 10)(2\ 9).$

(d) Find the parity of α .

By part (c) of the problem, α can be written as a product of 8 transpositions, hence its parity is even.

5. (a) How many permutations of order 5 are there in S_5 ?

Solution.

$$\frac{5!}{5} = 4 \times 3 \times 2 \times 1 = 24$$

(b) How many permutations of order 5 are there in S_6 ?

Solution. If a permutation of six elements has order five, it should fix one element and permute the other five. So there are

$$\binom{6}{5} \cdot \frac{5!}{5} = 6 \times (4 \times 3 \times 2 \times 1) = 6 \times 24 = 144$$

permutations in S_6 which have order five. The expression in the parentheses is from Part (a) of the problem.

- **6.** Find permutations α and β such that:
 - (a) $|\alpha| = 2$, $|\beta| = 2$, and $|\alpha\beta| = 2$.

Sample solution. Let $\alpha = (1 \ 2)$, $\beta = (3 \ 4)$. Then $\alpha\beta = (1 \ 2)(3 \ 4)$. About the order of $\alpha\beta$, since it's the product of cycles on two disjoint sets, $|\alpha\beta| = \operatorname{lcm}(|\alpha|, |\beta|) = 2$.

(b) $|\alpha| = 2$, $|\beta| = 2$, and $|\alpha\beta| = 3$.

Sample solution. Let $\alpha = (1 \ 2), \beta = (2 \ 3)$. Then $\alpha\beta = (1 \ 2 \ 3)$.

(c) $|\alpha| = 2$, $|\beta| = 4$, and $|\alpha\beta| = 4$.

Sample solution. Let $\alpha = (1 \ 2), \beta = (3 \ 4 \ 5 \ 6)$. Then $\alpha\beta = (1 \ 2)(3 \ 4 \ 5 \ 6)$.