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MATH 3175 **Group Theory** **Fall 2010**

Practice Quiz 6

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1. Let H be set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, with $a, b, d \in \mathbb{R}$ and $ad \neq 0$.

(a) Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(b) Is H a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$?

2. Let $H = \{(1), (12)(34)\}$.

(a) Show that H is a subgroup of A_4.

(b) What is the index of H in A_4?

(c) Is H a normal subgroup of A_4?

3. Let $G = U(32)$, and $H = \{1, 31\}$. Show that the quotient group G/H is isomorphic to $\mathbb{Z}_8$.

4. Let $G = \mathbb{Z}_4 \oplus U(4)$, and consider the subgroups $H = \langle (2,3) \rangle$ and $K = \langle (2,1) \rangle$.

(a) List the elements of G/H, and compute the Cayley table for this group. What is the isomorphism type of G/H?

(b) List the elements of G/K, and compute the Cayley table for this group. What is the isomorphism type of G/K?

(c) Are the groups G/H and G/K isomorphic?

5. Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, and consider the subgroups $H = \{(0,0), (2,0), (0,2), (2,2)\}$ and $K = \langle (1,2) \rangle$. Identify the following groups (as direct products of cyclic groups of prime order):

(a) H and G/H.

(b) K and G/K.

6. Give an example of a group G and a normal subgroup $H \triangleleft G$ such that both H and G/H are abelian, yet G is not abelian.

7. Let $\mathbb{Z}$ be the additive group of integers, and let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = 8x$.

(a) Show that f is a homomorphism.

(b) Find $\ker(f)$.

(c) Find $\mathrm{im}(f)$.

8. Let $\phi\colon G \to H$ and $\psi\colon H \to K$ be two homomorphisms.

(a) Show that $\psi \circ \phi\colon G \to K$ is a homomorphism.

(b) Show that $\ker(\phi)$ is a normal subgroup of $\ker(\psi \circ \phi)$.

9. Let G and H be two groups, and consider the map $p\colon G \oplus H \to H$ given by $p(g, h) = h$.

(a) Show that p is a homomorphism.

(b) What is $\ker(p)$? What is $\mathrm{im}(p)$?

(c) What does the First Isomorphism Theorem say in this situation?

10. Let $\phi\colon D_n \to \mathbb{Z}_2$ be the map given by

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is a rotation,} \\ 1 & \text{if } x \text{ is a reflection.} \end{cases}$$

(a) Show that ϕ is a homomorphism.

(b) What is $\ker(p)$? What is $\mathrm{im}(p)$?

11. Suppose $\phi\colon \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a homomorphism with $\phi(7) = 6$.

(a) Determine $\phi(x)$, for all $x \in \mathbb{Z}_{50}$.

(b) What is $\ker(\phi)$? What is $\mathrm{im}(\phi)$?

(c) What is $\phi^{-1}(3)$?

12. Show that there is no homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.