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MATH 3175 **Group Theory** **Fall 2010**

Solutions to Practice Quiz 6

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1. Let H be set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, with $a, b, d \in \mathbb{R}$ and $ad \neq 0$.

(a) Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

By definition, H is the set of nonsingular (invertible) upper triangular matrices, a subset of $\mathrm{GL}_2(\mathbb{R})$. The identity matrix I is in H. And H is closed under matrix multiplication: the product of nonsingular upper triangular matrices is nonsingular and upper triangular. Also, it is closed under taking inverses: the inverse of $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is $\begin{bmatrix} a^{-1} & -a^{-1}d^{-1}b \\ 0 & d^{-1} \end{bmatrix}$. Hence, it is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(b) Is H a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$?

No. For any $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in H$ and $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$, we have

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}^{-1} = \frac{1}{wx - yz} \begin{bmatrix} awx - bzx - dyz & bx^2 - ayx + dyx \\ -bz^2 + awz - dwz & dwx + bzx - ayz \end{bmatrix}.$$

In general, this matrix is not in H. **(To complete the answer, you should give a specific example.)**

2. Let $H = \{(1), (12)(34)\}$.

(a) Show that H is a subgroup of A_4.

H is generated by the permutation $(12)(34)$, which has order 2.

(b) What is the index of H in A_4?

$$\frac{|A_4|}{|H|} = \frac{12}{2} = 6.$$

(c) Is H a normal subgroup of A_4?

No. For $(321) \in A_4$ with inverse (123), we have

$$(321) \cdot (12)(34) \cdot (123) = (13)(24),$$

which is not in H.

3. Let $G = U(32)$, and $H = \{1, 31\}$. Show that the quotient group G/H is isomorphic to $\mathbb{Z}_8$.

Define a surjective homomorphism $\phi\colon G \to \mathbb{Z}_8$ by sending $1 \mapsto 0$, $3 \mapsto 1$, $9 \mapsto 2$, $27 \mapsto 3$, $17 \mapsto 4$, $19 \mapsto 5$, $25 \mapsto 6$, $11 \mapsto 7$, $7 \mapsto 2$, $23 \mapsto 6$, etc., and also $31 \mapsto 0$. You should check now that ϕ has kernel H. Thus, by the First Isomorphism Theorem, $G/H \cong \mathbb{Z}_8$.

4. Let $G = \mathbb{Z}_4 \oplus U(4)$, and consider the subgroups $H = \langle(2, 3)\rangle$ and $K = \langle(2, 1)\rangle$.

(a) List the elements of G/H, and compute the Cayley table for this group. What is the isomorphism type of G/H?

$G/H \cong \mathbb{Z}_4$.

(b) List the elements of G/K, and compute the Cayley table for this group. What is the isomorphism type of G/K?

$G/K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(c) Are the groups G/H and G/K isomorphic?

No.

5. Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, and consider the subgroups $H = \{(0, 0), (2, 0), (0, 2), (2, 2)\}$ and $K = \langle(1, 2)\rangle$. Identify the following groups (as direct products of cyclic groups of prime order):

(a) H and G/H.

Clearly, $H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$; indeed, the assignment $(0, 1) \mapsto (0, 1)$ and $(1, 0) \mapsto (1, 0)$ defines a homomorphism from G onto $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, with kernel H.

(b) K and G/K.

By definition, K is cyclic; since its generator, $(1, 2)$, has order 4, we have $K \cong \mathbb{Z}_4$. On the other hand, $G/K \cong \mathbb{Z}_4$, which can be seen by sending $(0, 1)$ and $(1, 0)$ to 1 and 2, respectively. This defines a homomorphism from G onto $\mathbb{Z}_4$, with kernel K.

6. Give an example of a group G and a normal subgroup $H \triangleleft G$ such that both H and G/H are abelian, yet G is not abelian.

Take $G = D_n$, with $n \geq 3$, and H the subgroup of rotations. (See Problem 10.) Then $H \cong \mathbb{Z}_n$ and $G/H \cong \mathbb{Z}_2$, but $G = D_n$ is not abelian.

7. Let $\mathbb{Z}$ be the additive group of integers, and let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = 8x$.

(a) Show that f is a homomorphism.

$f(x+y) = 8(x+y) = 8x + 8y = f(x) + f(y)$.

(b) Find $\ker(f)$.

$\ker(f) = \{0\}$

(c) Find $\operatorname{im}(f)$.

$\operatorname{im}(f) = 8\mathbb{Z} = \{8k | k \in \mathbb{Z}\}$

8. Let $\phi\colon G \to H$ and $\psi\colon H \to K$ be two homomorphisms.

(a) Show that $\psi \circ \phi\colon G \to K$ is a homomorphism.

We have:

$$\psi \circ \phi(g_1g_2) = \psi(\phi(g_1g_2)) = \psi(\phi(g_1)\phi(g_2)) = \psi \circ \phi(g_1)\psi \circ \phi(g_2).$$

(b) Show that $\ker(\phi)$ is a normal subgroup of $\ker(\psi \circ \phi)$.

For any $h \in \ker(\phi)$ and $g \in \ker(\psi \circ \phi)$, the conjugate ghg^{-1} is in $\ker(\phi)$:

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = e.$$

9. Let G and H be two groups, and consider the map $p\colon G \oplus H \to H$ given by $p(g,h) = h$.

(a) Show that p is a homomorphism.

We have:

$$p((g_1,h_1)(g_2,h_2)) = p(g_1g_2, h_1h_2) = h_1h_2 = p(g_1,h_1)p(g_2,h_2).$$

(b) What is $\ker(p)$? What is $\operatorname{im}(p)$?

$\ker(p) = G$, $\operatorname{im}(p) = H$.

(c) What does the First Isomorphism Theorem say in this situation?

View G as a subgroup, actually a normal subgroup, of $G \oplus H$. Then the quotient group, $(G \oplus H)/G$, is isomorphic to H.

10. Let $\phi\colon D_n \to \mathbb{Z}_2$ be the map given by

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is a rotation,} \\ 1 & \text{if } x \text{ is a reflection.} \end{cases}$$

(a) Show that ϕ is a homomorphism.

The product of two reflections is a rotation around the intersection point of the two reflection axes; the product of a reflection and a rotation is a reflection; and the product of two rotations is again a rotation.

(b) What is $\ker(p)$? What is $\operatorname{im}(p)$?

$\ker(p) \cong \mathbb{Z}_n$ is the cyclic subgroup generated by a rotation through $\frac{360}{n}$ degrees.

$\operatorname{im}(p) = \mathbb{Z}_2$.

11. Suppose $\phi\colon \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a homomorphism with $\phi(7) = 6$.

(a) Determine $\phi(x)$, for all $x \in \mathbb{Z}_{50}$.

$\phi(x) = 3x \mod 15$

(b) What is $\ker(\phi)$? What is $\operatorname{im}(\phi)$?

$\ker(\phi) = \{0, 5, 10, 15, \dots, 45\}$, while $\operatorname{im}(\phi) = \{0, 3, 6, 9, 12\}$

(c) What is $\phi^{-1}(3)$?

$\phi^{-1}(3) = 1 + \ker(\phi) = \{1, 6, 11, 16, \dots, 46\}$

12. Show that there is no homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Suppose there is a surjective homomorphism $\phi\colon \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \oplus \mathbb{Z}_4$. By the First Isomorphism Theorem,

$$\mathbb{Z}_8 \oplus \mathbb{Z}_2 / \ker(\phi) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4.$$

Thus,

$$|\ker(\phi)| = \frac{|\mathbb{Z}_8 \oplus \mathbb{Z}_2|}{|\mathbb{Z}_4 \oplus \mathbb{Z}_4|} = \frac{16}{16} = 1.$$

Hence, the kernel is trivial, i.e., $\ker \phi = \{(0,0)\}$. So ϕ is actually an isomorphism. But $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ has an element of order 8, while $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ does not. Contradiction.