1. Write down all the automorphisms of the group $\mathbb{Z}_5$.

The automorphisms are $\phi_k : \mathbb{Z}_5 \to \mathbb{Z}_5$, with $\phi_k(x) = kx$, for $k = 1, 2, 3, 4$.

2. Let $\mathbb{R}^+$ be the multiplicative group of positive real numbers. Show that the map $x \mapsto \sqrt[3]{x}$ is an automorphism of $\mathbb{R}^+$.

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the function given by $\phi(x) = \sqrt[3]{x}$. We need to show that $\phi$ is a bijection, and a homomorphism.

- $\phi$ injective. Suppose $\sqrt[3]{x} = \sqrt[3]{y}$. Taking cubes on both sides, we get $x = y$.
- $\phi$ surjective. Let $y \in \mathbb{R}^+$. Then $y = \phi(x)$, where $x = y^3$.
- $\phi$ a homomorphism. Compute: $\phi(x \cdot y) = \sqrt[3]{x \cdot y} = \sqrt[3]{x} \cdot \sqrt[3]{y} = \phi(x) \cdot \phi(y)$.

3. Show that the map $x \mapsto e^x$ is an isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, \cdot)$.

Let $\phi : \mathbb{R} \to \mathbb{R}^+$ be the function given by $\phi(x) = e^x$. We need to show that $\phi$ is a bijection, and a homomorphism.

- $\phi$ injective. Suppose $e^x = e^y$. Taking natural logs on both sides, we get $x = y$.
- $\phi$ surjective. Let $y \in \mathbb{R}^+$. Then $y = \phi(x)$, where $x = \log y$.
- $\phi$ a homomorphism. Compute: $\phi(x + y) = e^{x+y} = e^x \cdot e^y = \phi(x) \cdot \phi(y)$.

4. For each the following pair of groups, decide whether they are isomorphic or not. In each case, give a brief reason why.

(a) $U(5)$ and $U(10)$.
   Yes. They are both cyclic of order 4.

(b) $U(8)$ and $\mathbb{Z}_4$.
   No. $U(8)$ doesn’t has an element of order 4, but $\mathbb{Z}_4$ does.

(c) $U(10)$ and $\mathbb{Z}_4$.
   Yes. They are both cyclic of order 4.

(d) $S_3$ and $\mathbb{Z}_6$.
   No. $S_3$ is not abelian, but $\mathbb{Z}_6$ is.

(e) $S_3$ and $D_3$.
   Yes. They are both the permutation group of the three vertices of a triangle.

(f) $A_4$ and $D_6$.
   No. $A_4$ doesn’t has an element of order 6, but $D_6$ does.
5. Let $\phi: G \rightarrow H$ be an isomorphism between two groups. Suppose $G$ is abelian. Show that $H$ is also abelian.

Let $x, y \in H$. Since $\phi$ is a surjection, there are elements $a, b \in G$ such that $x = \phi(a)$ and $y = \phi(b)$. Since $\phi$ is a homomorphism, and $G$ is abelian, we have:

$$xy = \phi(a)\phi(b) = \phi(ab) = \phi(b)\phi(a) = yx.$$  

Hence, $H$ is abelian.

6. Let $g$ and $h$ be two elements in a group $G$, and let $\phi_g$ and $\phi_h$ be the corresponding inner automorphisms. Suppose $\phi_g = \phi_h$. Show that $h^{-1}g$ belongs to the center of $G$.

By definition, $\phi_g(a) = gag^{-1}$ and $\phi_h(a) = hah^{-1}$, for any $a \in G$. Thus, the condition $\phi_g = \phi_h$ means $gag^{-1} = hah^{-1}$, for all $a \in G$. Hence,

$$(h^{-1}g)a(h^{-1}g)^{-1} = h^{-1}gag^{-1}h = h^{-1}hah^{-1}h = a.$$  

Since this holds for any $a \in G$, we conclude that $h^{-1}g$ is in the center of $G$.


Using Lagrange’s Theorem, we get:

$$|G : K| = \frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = |G : H| \cdot |H : K|.$$  

8. Let $G$ be a group, and let $a$ be an element of order 24. How many left cosets of $\langle a^{10} \rangle$ in $\langle a \rangle$ are there? List all these cosets.

The order of $a^{10}$ is $24/\gcd(10, 24) = 24/2 = 12$. So the number of left cosets is $|\langle a \rangle| / |\langle a^{10} \rangle| = 24/12 = 2$. These cosets are:

$$\langle a^{10} \rangle = \{ e, a^{10}, a^{20}, a^{16}, a^2, \ldots \},$$

$$a \cdot \langle a^{10} \rangle = \{ a, a^{11}, a^{21}, a^{17}, a^3, \ldots \}.$$  

9. Let $D_4$ be dihedral group of order 8 (the group of symmetries of the square), let $H = \langle R_1 \rangle$ be the subgroup generated by a counter-clockwise rotation by 90°, and let $K = \langle S_0 \rangle$ be the subgroup generated by a reflection across the horizontal axis.

(a) Write down all the left cosets of $H$ in $D_4$.

$H, S_0 H$

(b) Write down all the right cosets of $H$ in $D_4$.

$H, HS_0$

(c) Write down all the left cosets of $K$ in $D_4$.

$K, R_1 K, R_2^3 K, R_1^3 K$

(d) Write down all the right cosets of $K$ in $D_4$.

$K, KR_1, KR_2^3, KR_1^3$

(e) Compute the indices $|D_4 : H|$ and $|D_4 : K|$.

2, 4
10. Let $S_4$ be the group of permutations of the set $\{1, 2, 3, 4\}$, and let $A_4$ the subgroup of even permutations.

(a) Write down all the left cosets of $A_4$ in $S_4$.
   $A_4$, $(1, 2)A_4$

(b) Write down all the right cosets of $A_4$ in $S_4$.
   $A_4$, $A_4(1, 2)$

(c) What is the index of $A_4$ in $S_4$?
   $|S_4 : A_4| = |S_4| / |A_4| = 4!/(4!/2) = 2.$

11. Suppose a group contains elements of orders 1 through 9. What is the minimum possible order of the group?

   The order of the group is divisible by 8, 9, 5 and 7. So it is at least $\text{lcm}(8, 9, 5, 7) = 8 \times 9 \times 5 \times 7 = 2,520$. And $\mathbb{Z}_{2,520}$ satisfies the stated condition. So 2,520 is the minimum possible order for such a group.

12. Suppose $K$ is a subgroup of $H$, and $H$ is a subgroup of $G$. If $|K| = 30$ and $|G| = 300$, what are the possible values for $|H|$?

   By Lagrange’s Theorem, we have: $|K| = 30$ divides $|H|$ and $|H|$ divides $|G| = 300$. So the possible values of $|H|$ are 30, 60, 150, and 300.

13. Suppose $|G| = 21$, and $G$ has precisely one subgroup of order 3, and one subgroup of order 7. Show that $G$ is cyclic.

   Suppose the subgroup of order 3 is $H$, and the one of order 7 is $K$, then there are $1 + 2 + 6 = 9$ elements in $H \cup K$. Choose an element $a$ from $G \setminus (H \cup K)$. Then the order of $a$ must be 21. Thus $G = \langle a \rangle$, and so $G$ is cyclic.

14. Let $G = \{(1), (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}$. For each integer $i$ from 1 to 4, find the stabilizer of $i$ and the orbit of $i$.

   Answer:
   
   \[
   \begin{array}{ccc}
   i & \text{Stab}_G i & \text{orbit of } i \\
   1 & (1), (24) & \{1, 2, 3, 4\} \\
   2 & (2), (13) & \{1, 2, 3, 4\} \\
   3 & (1), (24) & \{1, 2, 3, 4\} \\
   4 & (1), (13) & \{1, 2, 3, 4\}
   \end{array}
   \]