Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Fall 2010**

Some solutions to the problems on Practice Quiz 3

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1. (a) Find the subgroup lattice of $\mathbb{Z}_{36}$.

Subgroups: $\langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 4\rangle, \langle 6\rangle, \langle 9\rangle, \langle 12\rangle, \langle 18\rangle, \langle 0\rangle$
You should draw now the inclusions between these various subgroups.

(b) Make a table with all the elements of $\mathbb{Z}_{36}$, grouped according to their orders.

Corresponding orders: $36, 18, 12, 9, 6, 4, 3, 2, 1$.

(c) What are all the possible orders, and how many elements of each possible order are there?

You should list all elements having each possible order, and count them.

2. (a) List of the elements of $\mathbb{Z}_{40}$ that have order 10.

$4, 12, 28, 36$.

(b) Suppose $|x| = 10$. List of the elements of $\langle x\rangle$ that have order 10.

x, x^3, x^7, x^9.

3. Let G be a group, and H a subgroup of G. For any fixed $x \in G$, define the *conjugate* of H by x to be

$$xHx^{-1} = \{xhx^{-1} \mid h \in H\}.$$

Show that xHx^{-1} is a subgroup of G.

- $e \in xHx^{-1}$, since $e = xex^{-1} \in xHx^{-1}$.
- Let $a, b \in xHx^{-1}$. Then $a = xhx^{-1}$ and $b = xkx^{-1}$, for some $h, k \in H$. Now compute:
$$ab^{-1} = (xhx^{-1})(xk^{-1}x^{-1}) = xh(x^{-1}x)k^{-1}x^{-1} = x(hk^{-1})x^{-1}.$$
But H is a subgroup of G, and so hk^{-1} belongs to H. Hence, $ab^{-1} = x(hk^{-1})x^{-1}$ belongs to xHx^{-1}.

4. Let G be a group, and H a subgroup of G. Define the *normalizer* of H to be

$$N(H) = \{x \in G \mid xHx^{-1} = H\}.$$

Show that $N(H)$ is a subgroup of G.

Similar method.

5. Done in class.

6. Similar problem.

7. (a) Find the conjugate of $(1234)(56)$ by $a = (25)$ in S_7.
$(25)(1234)(56)(25)^{-1} = (25)(1234)(56)(25) = (1534)(26)$

(b) Find the conjugate of $(1234)(56)$ by $a = (27)$ in S_7.
$(27)(1234)(56)(27)^{-1} = (27)(1234)(56)(27) = (1734)(56)$

(c) Find the conjugate of $(1234)(56)$ by $a = (37)$ in S_7.
$(37)(1234)(56)(37)^{-1} = (37)(1234)(56)(37) = (1274)(56)$

8. How many permutations of order 5 are there in S_7?

- A permutation in S_7 has order 5 if and only if its cycle type is $(5, 1, 1)$, i.e., it is a 5-cycle.
- To count such cycles, first count the number of ways to choose 5 numbers out of 7 (distinct) numbers: this is $\binom{7}{5} = \frac{7 \cdot 6}{2 \cdot 1} = 21$.
- Then count the number of permuting those 5 numbers (this will give the cycles of length 5), and divide this number by 5 (so as to account for the fact that one can cyclically permute the entries in a cycle). This gives $5!/5 = 120/5 = 24$.
- Finally, multiply the above two numbers, to get $21 \cdot 24 = 504$.
- Answer: there are precisely 504 permutations of degree 7 and order 5.

9. How many permutations of order 6 are there in S_{10}?

Proceeding as above, let us list all possibilities for cycle types in S_{10} which have order 6, i.e., those (decreasing) sequences of positive numbers $(\lambda_1, \dots, \lambda_r)$ such that
- $\lambda_1 + \cdots + \lambda_r = 10$, and
- $\operatorname{lcm}(\lambda_1, \dots, \lambda_r) = 6$,

together with the number of possibilities for each type. Each time a cycle of length m appears, and there are k numbers at our disposal, we should multiply by $\binom{k}{m} \cdot \frac{k!}{k}$. Also, if a cycle of length m gets repeated s times, we should divide by $s!$, which represents the number of ways to rearrange those s cycles (of a fixed order m).

$(6, 3, 1)$	$\binom{10}{6}\frac{6!}{6} \cdot \binom{4}{3}\frac{3!}{3} = 201,600$
$(6, 2, 2)$	$\binom{10}{6}\frac{6!}{6} \cdot \binom{4}{2}\frac{2!}{2} \cdot \binom{2}{2}\frac{2!}{2} \cdot \frac{1}{2!} = 75,600$
$(6, 2, 1, 1)$	$\binom{10}{6}\frac{6!}{6} \cdot \binom{4}{2}\frac{2!}{2} = 151,200$
$(6, 1, 1, 1, 1)$	$\binom{10}{6}\frac{6!}{6} = 25,200$
$(3, 3, 2, 2)$	$\binom{10}{3}\frac{3!}{3} \cdot \binom{7}{3}\frac{3!}{3} \cdot \frac{1}{2!} \cdot \binom{4}{2}\frac{2!}{2} \cdot \binom{2}{2}\frac{2!}{2} \cdot \frac{1}{2!} = 25,200$
$(3, 3, 2, 1, 1)$	$\binom{10}{3}\frac{3!}{3} \cdot \binom{7}{3}\frac{3!}{3} \cdot \frac{1}{2!} \cdot \binom{4}{2}\frac{2!}{2} = 50,400$
$(3, 2, 2, 2, 1)$	$\binom{10}{3}\frac{3!}{3} \cdot \binom{7}{2}\frac{2!}{2} \cdot \binom{5}{2}\frac{2!}{2} \cdot \binom{3}{2}\frac{2!}{2} \cdot \frac{1}{3!} = 25,200$
$(3, 2, 2, 1, 1, 1)$	$\binom{10}{3}\frac{3!}{3} \cdot \binom{7}{2}\frac{2!}{2} \cdot \binom{5}{2}\frac{2!}{2} \cdot \frac{1}{2!} = 25,200$
$(3, 2, 1, 1, 1, 1, 1)$	$\binom{10}{3}\frac{3!}{3} \cdot \binom{7}{2}\frac{2!}{2} = 5,040$

Answer: the number of permutations of order 6 in S_{10} is $584,640$.

10. Let α and β be two permutations in S_n.

(a) Show that $\alpha\beta\alpha^{-1}\beta^{-1}$ is an even permutation.

Write α as a product of r transpositions, and β as a products of s transpositions. Then α^{-1} can also be written as a product of r transpositions, and β^{-1} can also be written as a products of s transpositions. Hence, $\alpha\beta\alpha^{-1}\beta^{-1}$ can be written as a product of $r+s+r+s = 2(r+s)$ transpositions, which is an even number of transpositions. Thus, $\alpha\beta\alpha^{-1}\beta^{-1}$ is an even permutation.

(b) Show that $\alpha\beta$ is even if and only if α and β are both even, or both odd.

Similar argument.

11. Let $\beta \in S_7$, and suppose $\beta^4 = (2143567)$. Find β.

Note that β is a 7-cycle. Thus, β has order 7, and so

$$\beta = \beta^8 = (\beta^4)^2 = (2143567) \cdot (2143567) = (2457136)$$

12. Find permutations α and β such that:

(a) $|\alpha| = 2$, $|\beta| = 2$, and $|\alpha\beta| = 3$.

$$\alpha = (12), \quad \beta = (23), \quad \alpha\beta = (123).$$

(b) $|\alpha| = 3$, $|\beta| = 3$, and $|\alpha\beta| = 5$.

$$\alpha = (123), \quad \beta = (345), \quad \alpha\beta = (12345).$$