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MATH 3175 **Group Theory** **Fall 2010**

Practice Quiz 2

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1. Let G be a group and let H and K be subgroups of G.

(a) Is $H \cap K$ a subgroup of G?

(b) Is $H \cup K$ a subgroup of G?

In each case, give a reason why, or why not.

2. Let H be a subgroup of a group G, and let $C(H) = \{x \in G \mid xh = hx \text{ for all } h \in H\}$. Prove that $C(H)$ is a subgroup of G.

3. The *quaternion group* is the group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, with multiplication given by the rules

$$\begin{aligned} ij &= k, & ji &= -k, \\ jk &= i, & kj &= -i, \\ ki &= j, & ik &= -j. \end{aligned}$$

(a) Write down a Cayley table for Q.

(b) For each element $a \in Q$, find the order $|a|$.

(c) For each element $a \in Q$, find the centralizer $C(a)$.

(d) What is the center of Q?

4. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ be a matrix in $M_2(\mathbb{Z}_7)$.

(a) Prove that A is in $\mathrm{GL}_2(\mathbb{Z}_7)$.

(b) Find the order of A in $\mathrm{GL}_2(\mathbb{Z}_7)$.

5. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) If we view A as an element in $\mathrm{SL}_2(\mathbb{R})$, what is the order of A?

(b) If we view A as an element in $\mathrm{SL}_2(\mathbb{Z}_p)$, for p a prime, what is the order of A?

6. Let G be a group, and a an element of G.

(a) Show that $C(a) \subseteq C(a^k)$, for all positive integers k.

(b) Suppose a has order 5. Prove that $C(a) = C(a^3)$.

(c) Give an example where a has order 6, and $C(a) \neq C(a^3)$.

7. Let $G = \mathbb{Z}_{20}$. Is G cyclic? For each element $a \in G$, indicate the order $|a|$. What are the generators of G?

8. Let $G = U(20)$. Is G cyclic? For each element $a \in G$, indicate the order $|a|$.

9. Let $G = U(7)$. Is G cyclic? For each element $a \in G$, indicate the order $|a|$. What are the generators of G?

10. Let G be a group, and let $a \in G$. Show that $\langle a \rangle = \langle a^{-1} \rangle$.

11. Give an example of a non-cyclic group, all of whose proper subgroup are cyclic.

12. Let $G = \langle a \rangle$ be a group generated by an element a of order $|a| = 24$.

(a) Find each element of G which generates G.

(b) Find each element of G which generates the subgroup $\langle a^3 \rangle$.

(c) Write all the elements of the subgroup $\langle a^3 \rangle$.

(d) Find the order of a^{16}.