Name: ______________________

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MATH 3150 **Real Analysis** **Spring 2011**

Midterm Exam

Instructions: Write your name in the space provided. Calculators are permitted, but no notes are allowed. Each problem is worth 10 points .

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1. Let $a_1 = 0$, $a_2 = 1$, $a_3 = \sqrt{3}$, $\dots$, $a_n = \sqrt{1 + 2a_{n-1}}$.

(a) Show that the sequence $\{a_n\}$ is strictly increasing.

(b) Show that the sequence $\{a_n\}$ is bounded above.

(c) Show that the sequence $\{a_n\}$ is converging. Give a reason for your answer.

(d) Find $\lim\limits_{n\to\infty} a_n$.

2. Let $\{x_n\}$ be a sequence in a complete metric (X, d).

(a) Suppose $d(x_{n+1}, x_n) \leq 1/2^n$, for all $n \geq 1$. Show that $\{x_n\}$ converges.

(b) Suppose $d(x_{n+1}, x_n) \leq 1/n$, for all $n \geq 1$. Show by example that $\{x_n\}$ may *not* converge.

3. Let A and B be two non-empty subsets of $\mathbb{R}$, and write $A+B = \{x+y \mid x \in A \text{ and } y \in B\}$.

(a) Show that
$$\sup(A+B) = \sup(A) + \sup(B).$$

(b) Now suppose $A = \{x \in \mathbb{R} \mid x^2 < 2\}$ and $B = \{y \in \mathbb{R} \mid y < 3\}$. Compute:
$$\sup(A), \quad \sup(B), \quad \sup(A+B).$$

4. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ x \geq 0,\ y > 0\}$.

(a) Draw a picture of the set A.

(b) What is the interior of A? Is A an open subset of $\mathbb{R}^2$?

(c) What is the closure of A? Is A a closed subset of $\mathbb{R}^2$?

(d) What is the boundary of A?

5. Let (X, d) be a metric space, A a subset of X, and x a point in X. We say that:

- The point x is an *accumulation* point for A if every open set U containing x contains some point of A other than x.
- The point x is a *limit* point for A if every open set U containing x contains some point of A.

(a) Suppose x is a limit point for A, and $x \notin A$; then show that x is an accumulation point for A.

(b) Let $A = \{1/n \mid n \in \mathbb{N}\}$, viewed as a subset of $\mathbb{R}$.

- What are the limit points of A?
- What are the accumulation points of A?
- Does the set of limit points coincide with the set of accumulation points?

6. Decide whether each of the following series converges or not. In each case, indicate which test is used, and why that test yields the conclusion you are drawing.

(a)

$$\sum_{k=1}^{\infty} \frac{100^k}{k!}$$

(b)

$$\sum_{k=2}^{\infty} \frac{k}{\sqrt{k^4 - 1}}$$