The final exam will focus on the assigned material in the text from section 14 onward.

For the final exam you are allowed one two-sided page of notes on a standard, 8 $\frac{1}{2}$ by 11 inches, piece of paper. No additional notes or scratch paper are allowed. You may use the blank, unnumbered, pages on the back of each numbered page for your work if needed. If you do this, be sure to note on the numbered page where the reader should look for the continuation of your work on the problem.
Cellphones and laptops must be turned off and placed on the floor.
For credit you need to fully justify your response to each question. You can cite results in the text by indicating the result—for example, *since every bounded sequence contains a convergent subsequence, it follows that* ......

1. Suppose f is a continuous function defined on $\mathbb{R}$ with f' a nonnegative increasing function on $\mathbb{R}$ with $\lim_{x\to+\infty} f'(x) = +\infty$. Prove that f is uniformly continuous on the interval $(-\infty, a]$ for each $a \in \mathbb{R}$.

   Solution: Let $a \in \mathbb{R}$. It suffices to show that given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{for all } x, y \in (-\infty, a] \text{ and } |x - y| < \delta \tag{1}$$

   This can be done as follows. Given $\epsilon > 0$, set $\delta = \epsilon/f'(a)$. Note that $\epsilon/f'(a) > 0$ since $f'(a) > 0$. From the Mean Value theorem, it follows that $f(x) - f(y) = f'(c) \cdot (x - y)$ for some c between x and y, ad we have the following

$$\begin{aligned} |f(x) - f(y)| &= |f'(c) \cdot (x - y)| \\ &= f'(c)|x - y| \\ &\le f'(a)|x - y| \\ &\le f'(a) \cdot \frac{\epsilon}{f'(a)} = \epsilon \end{aligned}$$

   where the second line follows since $f\prime$ is nonnegative, and the third line follows since f' is increasing. This completes the proof of equation (1).

2. Suppose $f(x)$ is a function differentiable for all x in $[0, \infty)$, and $f'(x) \to 0$ as $x \to +\infty$. Let $g(x) = f(x + 1) - f(x)$. Using the mean value theorem, show that $\lim\limits_{x\to+\infty} g(x) = 0$.

   Solution: Given any $\epsilon > 0$, it suffices to show that there is an M such that

$$|g(x)| < \epsilon \quad \text{for all } x > M$$

   To do this, assume $\epsilon > 0$ has been given. Then since $\lim_{x\to+\infty} f'(x) = 0$, there is an M such that

$$|f'(c)| < \epsilon \quad \text{for all } c > M \tag{2}$$

   Then for any x we have from the Mean Value Theorem that

$$f(x + 1) - f(x) = f'(c) \cdot (x + 1 - x) = f'(c)$$

for some c between x and $x+1$. If $x > M$, then $c > M$ and we have

$$|g(x)| = |f(x+1) - f(x)| = |f'(c)| < \epsilon$$

which completes the argument that $\lim_{x\to+\infty} g(x) = 0$.

3. Let $g(x) = f(x^3)+x$ where $f\colon [0,1] \to \mathbb{R}$ is a differentiable function such that $f(0) = f(1)$. Show that there exists a point $c \in [0,1]$ such that $g'(c) = 1$.

   Solution: From the chain rule, we have that

   $$g'(x) = 3x^2 f'(x^3) + 1$$

   Thus, $g'(x) = 1$ iff $f'(x^3) = 0$. Since $f(0) = f(1)$, it follows from Rolle's Theorem that $f'(y) = 0$ for some $y \in (0,1)$. Note that for $y \in (0,1)$, we have $\sqrt[3]{y} \in (0,1)$, and hence we have $g'(c) = 1$ with $c = \sqrt[3]{y} \in (0,1)$.

4. (a) Let $p(t) = t^3 + at^2 + bt + c$ be a cubic polynomial with real coefficients $a, b, c \in \mathbb{R}$. Use the Intermediate Value Theorem to show that p has a real root, i.e., there exists $t_0 \in \mathbb{R}$ such that $p(t_0) = 0$.

   Solution: The polynomial $p(t)$ defines a continuous function, $p\colon \mathbb{R} \to \mathbb{R}$. For $t > 0$, we have that

   $$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} t^3[1 + a/t + b/t^2 + c/t^3] = \infty.$$

   Thus, there is a $t_1 > 0$ such that $p(t_1) > 0$.

   Similarly, for $t < 0$ we have that $\lim_{t\to-\infty} p(t) = -\infty$, and so there is a $t_2 < 0$ such that $p(t_2) < 0$.

   By the Intermediate Value Theorem, there exists a $t_0 \in (t_2, t_1)$ such that $p(t_0) = 0$.

   (b) What can you say about existence of real roots for a polynomial of arbitrary degree $k \in \mathbb{N}$?

   Solution: A similar argument shows that any polynomial $p(t)$ of *odd* degree $k = 2n+1$ must have a real root.

   On the other hand, if the polynomial $p(t)$ has *even* degree $k = 2n$, the previous argument breaks down, since in this case $\lim_{t\to\infty} p(t) = \lim_{t\to-\infty} p(t) = \infty$. The polynomial may or may not have a real root; for instance, $p(t) = t^2 - 1$ has real roots $t = \pm 1$, whereas $p(t) = t^2 + 1$ has no real roots.

5. Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that there exists some $x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$.

   Solution: If $f(a) = g(a)$ or $f(b) = g(b)$, then the result holds, so assume $f(a) > g(a)$ and $f(b) < g(b)$. Set $h(x) = f(x) - g(x)$, then h is continuous since a difference of continuous functions is continuous. Then we have $h(a) > 0$ and $h(b) < 0$ so it follows from the Intermediate Value Theorem that $h(x_0) = 0$ for some $x_0 \in (a,b)$ and the result follows since $h(x_0) = f(x_0) - g(x_0)$.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by

   $$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is continuous at $x = 0$, but discontinuous everywhere else.

Solution: When $x_0 = 0$, we have that $f(x_0) = 0$ and also $\lim_{x\to 0} f(x) = 0$; thus, f is continuous at $x_0 = 0$.

On the other hand, when $x_0 \neq 0$, we claim that $\lim_{x\to x_0} f(x)$ does not exist. Indeed, consider the following two cases:

- Let (x_n) be a sequence of rational numbers such that $x_n \to x_0$; then $f(x_n) = x_n$ and so $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x_0$.
- Let (x_n) be a sequence of irrational numbers such that $x_n \to x_0$; then $f(x_n) =$ and so $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$.

Since $x_0 \neq 0$, these two limits differ. This shows that $\lim_{x\to x_0} f(x)$ does not exist, which implies that f is not continuous at x_0.

7. Consider the sequences of functions $f_n\colon [-1,1] \to \mathbb{R}$ and $g_n\colon [0,1] \to \mathbb{R}$ given by $f_n(x) = x^n$ and $g_n(x) = x^n$.

   (a) Does either of these sequences converge? If it does, what is its limit? If it doesn't, why not?

   Solution: The first sequence does not converge, since $f_n(-1) = (-1)^n$, and so $\lim_{n\to\infty} f_n(-1)$ does not exist (the subsequence $(-1)^{2k}$ converges to $+1$, whereas the subsequence $(-1)^{2k+1}$ converges to -1).

   The second sequence converges: its limit is the function $g\colon [0,1] \to \mathbb{R}$ given by $g(x) = 0$ for $0 \le x < 1$ and $g(1) = 1$.

   (b) Does either of these sequences converge uniformly? Why or why not?

   Solution: The sequence (f_n) does not converge (pointwise); thus, it does not converge uniformly.

   Although the sequence (g_n) converges (pointwise), it does not converge uniformly. Indeed, the functions g_n are continuous, and $g_n \to g$, but g is not continuous (at $x = 1$), as it would be if the convergenvce would be uniform.

8. Let $-\infty < a < b < \infty$. Suppose $f\colon [a,b] \to \mathbb{R}$ is a continuous function, and let $F\colon [a,b] \to \mathbb{R}$ be a function differentiable on (a,b) such that $F'(x) = f(x)$.

   (a) Find the limit

$$\lim_{n\to\infty} \frac{b-a}{n}\left[f(a) + f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \cdots + f\left(a + \frac{(n-1)(b-a)}{n}\right)\right].$$

   Solution: Note that

$$\frac{b-a}{n}\left[f(a) + f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \cdots + f\left(a + \frac{(n-1)(b-a)}{n}\right)\right]$$

   is the Riemann sum for f given by the partition $P_n = \left\{a + \frac{i(b-a)}{n} : 0 \le i \le n\right\}$ with mesh equal to $1/n$ and by choosing to evaluate f at the left hand endpoint of each subinterval.

   Since f is continuous, it follows that f is integrable and since $\lim_{n\to\infty} \operatorname{mesh}(P_n) = 0$, it follows that the limit of Riemann sums given in the statement of the problem equals $\int_a^b f(t)\,dt$. From the First Fundamental Theorem of Calculus, it now follows that $\int_a^b f(t)\,dt = F(b) - F(a)$.

   Thus, the limit of the sums given in the statement of the problem equals $F(b) - F(a)$.

   (b) Compute the limit

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right).$$

   Solution: The limit of sums above is the special case of the limit of sums in part (a) with $F(x) = e^x, f(x) = e^x$, and $[a,b] = [0,1]$. From the result in part (a), we have that the limit of the sums is $F(1) - F(0) = e^1 - 1$.

9. Suppose $f\colon [0,1] \to [0,1]$ is a continuous function which maps $[0,1]$ into $[0,1]$.

   (a) Show that there exists a point $c \in [0,1]$ such that $f(c) = c$. (*Hint: Let* $g(x) = f(x) - x$.)

   Solution: Let $g\colon [0,1] \to \mathbb{R}$ be the function given by $g(x) = f(x) - x$. Since both f and the function $h(x) = x$ are continuous, the difference $g = f - h$ is also continuous.

   By assumption, $0 \le f(x) \le 1$ for all $x \in [0,1]$. Therefore, $g(0) = f(0) \ge 0$ and $g(1) = f(1) - 1 \le 0$.

   By the Intermediate Value Theorem, there exists a $c \in [0,1]$ such that $g(c) = 0$, and so $f(c) = c$.

   (b) Find the point c as in part (a) for $f(x) = \dfrac{x+1}{4}$.

   Solution: The equation $c = \dfrac{c+1}{4}$ has (unique) solution $c = 1/3$.

10. Let f be the function defined on $[-\pi, \pi]$ by

$$f(x) = \int_0^{x^2} e^{\sin t}\,dt$$

(a) Show that f is differentiable on $(-\pi, \pi)$. What is its derivative function?

Solution: The integrand function $e^{\sin t}$ is continuous on $\mathbb{R}$, therefore integrable on $[0, \pi^2]$ and by the Fundamental Theorem of Calculus

$$g(u) := \int_0^u e^{\sin t}\, dt$$

is differentiable on $u \in [0, \pi^2]$ and $g'(u) = e^{\sin u}$. Then we have $f(x) = g(x^2)$ is the composition of g and $u = x^2$, hence is differentiable on $x \in [-\pi, \pi]$ (corresponding to $u = x^2 \in [0, \pi^2]$).

Then we apply the chain rule to obtain

$$f'(x) = g'(x^2)(x^2)' = 2xe^{\sin x^2}$$

(b) Evaluate $f(0)$ and $f'(\sqrt{\pi}/2)$.

Solution: From part (a), we have $f(0) = g(0^2) = g(0) = \int_0^0 e^{\sin t}\, dt = 0$ and $f'(\sqrt{\pi/2}) = 2\sqrt{\pi/2}e^{\sin(\pi/2)} = 2\sqrt{\pi/2}e = e\sqrt{2\pi}$.