The final exam will focus on the assigned material in the text from section 14 onward.

For the final exam you are allowed one two-sided page of notes on a standard, 8 $\frac{1}{2}$ by 11 inches, piece of paper. No additional notes or scratch paper are allowed. You may use the blank, unnumbered, pages on the back of each numbered page for your work if needed. If you do this, be sure to note on the numbered page where the reader should look for the continuation of your work on the problem.
Cellphones and laptops must be turned off and placed on the floor.
For credit you need to fully justify your response to each question. You can cite results in the text by indicating the result—for example, *since every bounded sequence contains a convergent subsequence, it follows that* ......

1. Suppose f is a continuous function defined on $\mathbb{R}$ with f' a nonnegative increasing function on $\mathbb{R}$ with $\lim_{x\to+\infty} f'(x) = +\infty$. Prove that f is uniformly continuous on the interval $(-\infty, a]$ for each $a \in \mathbb{R}$.

2. Suppose $f(x)$ is a function differentiable for all x in $[0,\infty)$, and $f'(x) \to 0$ as $x \to +\infty$. Let $g(x) = f(x+1) - f(x)$. Using the mean value theorem, show that $\lim\limits_{x\to+\infty} g(x) = 0$.

3. Let $g(x) = f(x^3)+x$ where $f\colon [0,1] \to \mathbb{R}$ is a differentiable function such that $f(0) = f(1)$. Show that there exists a point $c \in [0,1]$ such that $g'(c) = 1$.

4. (a) Let $p(t) = t^3 + at^2 + bt + c$ be a cubic polynomial with real coefficients $a, b, c \in \mathbb{R}$. Use the Intermediate Value Theorem to show that p has a real root, i.e., there exists $t_0 \in \mathbb{R}$ such that $p(t_0) = 0$.

   (b) What can you say about existence of real roots for a polynomial of arbitrary degree $k \in \mathbb{N}$?

5. Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that there exists some $x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$
   Prove that f is continuous at $x = 0$, but discontinuous everywhere else.

7. Consider the sequences of functions $f_n\colon [-1,1] \to \mathbb{R}$ and $g_n\colon [0,1] \to \mathbb{R}$ given by $f_n(x) = x^n$ and $g_n(x) = x^n$.

   (a) Does either of these sequences converge? If it does, what is its limit? If it doesn't, why not?

   (b) Does either of these sequences converge uniformly? Why or why not?

8. Let $-\infty < a < b < \infty$. Suppose $f\colon [a,b] \to \mathbb{R}$ is a continuous function, and let $F\colon [a,b] \to \mathbb{R}$ be a function differentiable on (a,b) such that $F'(x) = f(x)$.
   (a) Find the limit

$$\lim_{n\to\infty} \frac{b-a}{n}\left[f(a) + f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \cdots + f\left(a + \frac{(n-1)(b-a)}{n}\right)\right].$$

   (b) Compute the limit

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right).$$

9. Suppose $f\colon [0,1] \to [0,1]$ is a continuous function which maps $[0,1]$ into $[0,1]$.
   (a) Show that there exists a point $c \in [0,1]$ such that $f(c) = c$. (*Hint: Let* $g(x) = f(x) - x$.)
   (b) Find the point c as in part (a) for $f(x) = \dfrac{x+1}{4}$.

10. Let f be the function defined on $[-\pi, \pi]$ by

$$f(x) = \int_0^{x^2} e^{\sin t}\, dt$$

   (a) Show that f is differentiable on $(-\pi, \pi)$. What is its derivative function?
   (b) Evaluate $f(0)$ and $f'(\sqrt{\pi}/2)$.