Math 3150 Final Exam Fall 2016

Name: ____

- The exam will last 2 hours.
- There are 8 problems.
- One single-sided sheet of theorems and definitions is allowed.
- Use the back sides of the test pages for scratch work, or if you need extra space.

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Total	

Problem 1. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2} \qquad \text{for } x \in \mathbb{R},\ n \in \mathbb{N}.$$

(a) (8 pts) Compute the derivative $f_n'(x)$. Then find the pointwise limit $g(x)$ of the sequence of derivatives $f_n'(x)$ as $n \to \infty$.

(b) (6 pts) Does the sequence of derivatives $f_n'(x)$ converge uniformly? Why, or why not?

Problem 2. Let $g(x) = f(x) + 2x$, where $f\colon [0,1] \to \mathbb{R}$ is a differentiable function which satisfies $f(0) = f(1)$.

(a) (8 pts) Show that there exists a point $c \in [0,1]$ such that $g'(c) = 2$.

(b) (6 pts) Find explicitly such a point $c \in [0,1]$ for $f(x) = x(x-1)$.

Problem 3 (10 pts)**.** Let X, Y and Z be subsets of $\mathbb{R}$. Suppose $f\colon X \to Y$ is a uniformly continuous function on X, and $g\colon Y \to Z$ is a uniformly continuous function on Y. Show that the composition $g \circ f\colon X \to Z$ defined by

$$g \circ f(x) = g\big(f(x)\big)$$

is uniformly continuous.

Problem 4 (12 pts)**.** Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^3+n}{n^3}$$

converges uniformly on any bounded interval $[0, M]$ for some $M > 0$, but doesn't converge uniformly on $[0, +\infty)$. (*Hint: Use the M-Test and Cauchy's criterion.*)

Problem 5 (3 pts each)**.** Determine whether each of the following sets is compact or not. Explain your reasons.

(a) $\left\{n + \dfrac{1}{n} : n \in \mathbb{N}\right\}$.

(b) $\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2\right\}$.

(c) $\left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$.

(d) $\left\{n \sin\left(\dfrac{1}{n}\right) : n \in \mathbb{N}\right\} \cup \{1\}$.

Problem 6 (12 pts)**.** Suppose the function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\big|f(x) - f(y)\big| \leq C|x-y|^2 \qquad \text{for all } x, y \in \mathbb{R},$$

for some $C > 0$. Show that f must be constant. (*Hint: First show that f is differentiable.*)

Problem 7. Let $-\infty < a < b < \infty$. Suppose $f\colon [a,b] \to \mathbb{R}$ is a continuous function, and let $F\colon [a,b] \to \mathbb{R}$ be a function differentiable on (a,b) such that $F'(x) = f(x)$.

(a) (7 pts) Find the limit

$$\lim_{n\to\infty} \frac{b-a}{n}\left[f(a) + f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \cdots + f\left(a + \frac{(n-1)(b-a)}{n}\right)\right]$$

(b) (7 pts) Compute the limit

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right)$$

Problem 8. Suppose $f\colon [0,1] \to [0,1]$ is a continuous function which maps $[0,1]$ into $[0,1]$.

(a) (8 pts) Show that there exists a point $c \in [0,1]$ such that $f(c) = c$. (*Hint: Let* $g(x) = f(x) - x$.)

(b) (4 pts) Find the point c as in part (a) for $f(x) = \dfrac{x+1}{4}$.