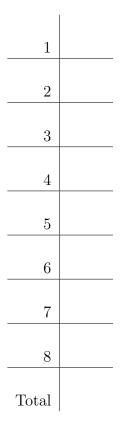
Math 3150 Final Exam Fall 2016

Name: _____

- The exam will last 2 hours.
- There are 8 problems.
- One single-sided sheet of theorems and definitions is allowed.
- Use the back sides of the test pages for scratch work, or if you need extra space.



Problem 1. Consider the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2}$$
 for $x \in \mathbb{R}, n \in \mathbb{N}$.

(a) (8 pts) Compute the derivative $f'_n(x)$. Then find the pointwise limit g(x) of the sequence of derivatives $f'_n(x)$ as $n \to \infty$.

(b) (6 pts) Does the sequence of derivatives $f'_n(x)$ converge uniformly? Why, or why not?

Problem 2. Let g(x) = f(x) + 2x, where $f: [0,1] \to \mathbb{R}$ is a differentiable function which satisfies f(0) = f(1).

(a) (8 pts) Show that there exists a point $c \in [0, 1]$ such that g'(c) = 2.

(b) (6 pts) Find explicitly such a point $c \in [0, 1]$ for f(x) = x(x - 1).

Problem 3 (10 pts). Let X, Y and Z be subsets of \mathbb{R} . Suppose $f: X \to Y$ is a uniformly continuous function on X, and $g: Y \to Z$ is a uniformly continuous function on Y. Show that the composition $g \circ f: X \to Z$ defined by

$$g \circ f(x) = g(f(x))$$

is uniformly continuous.

Problem 4 (12 pts). Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \, \frac{x^3 + n}{n^3}$$

converges uniformly on any bounded interval [0, M] for some M > 0, but doesn't converge uniformly on $[0, +\infty)$. (*Hint: Use the M-Test and Cauchy's criterion.*)

Problem 5 (3 pts each). Determine whether each of the following sets is compact or not. Explain your reasons.

(a)
$$\left\{ n + \frac{1}{n} : n \in \mathbb{N} \right\}$$
.

(b)
$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 2\}.$$

(c)
$$\left\{\frac{1}{n}: n \in \mathbb{N}\right\}$$
.

(d)
$$\left\{ n \sin\left(\frac{1}{n}\right) : n \in \mathbb{N} \right\} \cup \{1\}.$$

Problem 6 (12 pts). Suppose the function $f \colon \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \le C|x - y|^2$$
 for all $x, y \in \mathbb{R}$,

for some C > 0. Show that f must be constant. (*Hint: First show that f is differentiable.*)

Problem 7. Let $-\infty < a < b < \infty$. Suppose $f: [a, b] \to \mathbb{R}$ is a continuous function, and let $F: [a, b] \to \mathbb{R}$ be a function differentiable on (a, b) such that F'(x) = f(x).

(a) (7 pts) Find the limit

$$\lim_{n \to \infty} \frac{b-a}{n} \left[f(a) + f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2(b-a)}{n}\right) + \dots + f\left(a + \frac{(n-1)(b-a)}{n}\right) \right]$$

(b) (7 pts) Compute the limit

$$\lim_{n \to \infty} \frac{1}{n} \left(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \dots + e^{\frac{n-1}{n}} \right)$$

Problem 8. Suppose $f: [0,1] \rightarrow [0,1]$ is a continuous function which maps [0,1] into [0,1].

(a) (8 pts) Show that there exists a point $c \in [0,1]$ such that f(c) = c. (*Hint: Let* g(x) = f(x) - x.)

(b) (4 pts) Find the point c as in part (a) for $f(x) = \frac{x+1}{4}$.