Name: ______________________

Prof. Alexandru Suciu

MATH 3150 | **Real Analysis** | **Fall 2011**

Midterm Exam

Instructions: Write your name in the space provided. Calculators are permitted, but no notes are allowed. Each problem is worth 10 points (with a bonus question worth 5 points).

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1. Let $a_1 = 1$, $a_2 = 3$, $\dots$, $a_n = \sqrt{7 + 2a_{n-1}}$.

(a) Show that the sequence $\{a_n\}_{n\geq 1}$ is strictly increasing.

(b) Show that the sequence $\{a_n\}_{n\geq 1}$ is bounded above.

(c) Show that the sequence $\{a_n\}_{n\geq 1}$ is converging. Give a reason for your answer.

(d) Find $\lim\limits_{n\to\infty} a_n$.

2. Let $\{x_n\}_{n\geq 1}$ be a sequence in a complete metric space (X, d).

(a) Suppose $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})/2$, for all $n \geq 2$. Show that $\{x_n\}$ converges.

(b) Suppose $d(x_{n+1}, x_n) \leq 1/\sqrt{n}$, for all $n \geq 1$. Show by example that $\{x_n\}$ may *not* converge.

3. (a) Let $S = \{(x, y) \in \mathbb{R}^2 \mid x + y > 1\}$, and let $A = \{d((x, y), (0, 0)) \mid (x, y) \in S\}$. Find $\inf(A)$.

(b) Let $A = \{x \in \mathbb{R} \mid x^2 < 3\}$ and $B = \{y \in \mathbb{R} \mid y < 2\}$. Find $\sup(A)$, $\sup(B)$, and $\sup(A + B)$.

4. Let (X, d) be a metric space and A a subset of X.

(a) Define the sets $\mathrm{int}(A)$, $\mathrm{cl}(A)$, and $\mathrm{bd}(A)$, i.e., the interior, the closure, and the boundary of A.

(b) Recall that $\mathrm{cl}(A) = A \cup A'$, where A' denotes the set of points $x \in X$ having the property that every open set U containing x also contains some point of A other than x. Use this information to show that:

$$x \in \mathrm{cl}(A) \iff D(x, \epsilon) \cap A \neq \emptyset, \ \forall \epsilon > 0,$$

(c) (Bonus question: 5 points) Use part (b) to show that $\mathrm{int}(A) = A \setminus \mathrm{bd}(A)$.

5. Let
$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ x < 0,\ y \leq 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid y = x,\ 0 < x < 1\}.$

(a) Draw a picture of the set A.

(b) What is the interior of A? Is A an open subset of $\mathbb{R}^2$?

(c) What is the closure of A? Is A a closed subset of $\mathbb{R}^2$?

(d) What is the boundary of A?

6. Decide whether each of the following series converges or not. In each case, indicate which test is used, and why that test yields the conclusion you are drawing.

(a)

$$\sum_{n=1}^{\infty} \frac{2n-1}{n^3+1}$$

(b)

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3 \cdot \log(n+1)}$$