

## Handout 1: Inverse images and direct images

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Let  $f: A \rightarrow B$  be a function, and let  $U \subset B$  be a subset. The *inverse image* (or, *preimage*) of  $U$  is the set  $f^{-1}(U) \subset A$  consisting of all elements  $a \in A$  such that  $f(a) \in U$ .

The inverse image commutes with all set operations: For any collection  $\{U_i\}_{i \in I}$  of subsets of  $B$ , we have the following identities for

(1) Unions:

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i)$$

(2) Intersections:

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i)$$

and for any subsets  $U$  and  $V$  of  $B$ , we have identities for

(3) Complements:

$$(f^{-1}(U))^c = f^{-1}(U^c)$$

(4) Set differences:

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

(5) Symmetric differences:

$$f^{-1}(U \Delta V) = f^{-1}(U) \Delta f^{-1}(V)$$

In addition, for  $X \subset A$  and  $Y \subset B$ , the inverse image satisfies the miscellaneous identities

(6)  $(f|_X)^{-1}(Y) = X \cap f^{-1}(Y)$

(7)  $f(f^{-1}(Y)) = Y \cap f(A)$

(8)  $X \subset f^{-1}(f(X))$ , with equality if  $f$  is injective.

Let  $f: A \rightarrow B$  be a function, and let  $U \subset A$  be a subset. The *direct image* (or, simply, *image*) of  $U$  is the set  $f(U) \subset B$  consisting of all elements of  $B$  which equal  $f(u)$  for some  $u \in U$ .

Direct images satisfy the following properties:

- (1) Unions: For any collection  $\{U_i\}_{i \in I}$  of subsets of  $A$ ,

$$f\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f(U_i).$$

- (2) Intersections: For any collection  $\{U_i\}_{i \in I}$  of subsets of  $A$ ,

$$f\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f(U_i).$$

- (3) Set difference: For any  $U, V \subset A$ ,

$$f(V \setminus U) \supset f(V) \setminus f(U).$$

In particular, the complement of  $U$  satisfies  $f(U^c) \supset f(A) \setminus f(U)$ .

- (4) Subsets: If  $U \subset V \subset A$ , then  $f(U) \subset f(V) \subset B$ .  
(5) Inverse image of a direct image: For any  $U \subset A$ ,

$$f^{-1}(f(U)) \supset U$$

with equality if  $f$  is injective.

- (6) Direct image of an inverse image: For any  $V \subset B$ ,

$$f(f^{-1}(V)) \subset V$$

with equality if  $f$  is surjective.