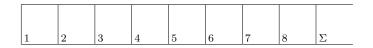
Name:



## NORTHEASTERN UNIVERSITY DEPARTMENT OF MATHEMATICS

**MATH 3150** 

Real Analysis Final Exam Fall 2011

Put your name in the blanks above. Calculators are permitted. A single sheet of theorems and definitions is allowed. Provide explanations for all your answers. If needed, use the back of the page for additional space.

**1.** Let  $f: [2,3] \to \mathbb{R}$  be a function, continuous on [2,3], and differentiable on (2,3). Suppose that f(2) = 6 and f(3) = 9. Show that, for some point  $x_0 \in (2,3)$ , the tangent line to the graph of f at  $x_0$  passes through the origin. Illustrate the result with a sketch. **2.** (a) Give an example of a connected subset  $A \subset \mathbb{R}$  such that its complement,  $\mathbb{R} \setminus A$ , is also connected.

(b) Give an example of a connected subset  $A \subset \mathbb{R}$  such that its complement,  $\mathbb{R} \setminus A$ , *not* connected.

(c) Give an example of a family of open subsets  $\{U_i\}_{i \in I}$  of  $\mathbb{R}$  such that their intersection,  $\bigcap_{i \in I} U_i$  is *not* open.

(d) Give an example of a family of closed subsets  $\{C_i\}_{i \in I}$  of  $\mathbb{R}$  such that their union,  $\bigcup_{i \in I} C_i$  is *not* closed.

- **3.** Let  $f : \mathbb{R} \to \mathbb{R}$  be a continuous function, and let  $G = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$  be its graph.
  - (a) Show that G is a closed subset of  $\mathbb{R}^2$ .

(b) Show that G is a connected subset of  $\mathbb{R}^2$ .

(c) Is G a compact subset of  $\mathbb{R}^2$ ? Why, or why not?

4. Consider the following subset of the plane:

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + y^2 \le 1 \right\}.$$

(a) Sketch the set A.

(b) Is A a closed subset of  $\mathbb{R}^2$ ? Why, or why not?

(c) Is A a bounded subset of  $\mathbb{R}^2$ ? Why, or why not?

(d) Is A a compact subset of  $\mathbb{R}^2$ ? Why, or why not?

- **5.** Let  $f \colon \mathbb{R} \to \mathbb{R}$  and  $g \colon \mathbb{R} \to \mathbb{R}$  be two Lipschitz functions.
  - (a) Show that the composition of the two functions,  $g \circ f \colon \mathbb{R} \to \mathbb{R}$ , is also a Lipschitz function.

(b) Suppose now that both f and g are bounded functions. Show that the product of the two functions,  $f \cdot g \colon \mathbb{R} \to \mathbb{R}$ , is also a Lipschitz function.

(c) (Bonus) Give an example where  $f \cdot g$  is *not* a Lipschitz function.

**6.** Consider the function  $f : \mathbb{R} \to \mathbb{R}$  defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{2\pi x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Show that f is continuous.

(b) Show that the restriction of f to the interval [-1, 1] is uniformly continuous.

(c) Show that f is not differentiable at x = 0.

7. Consider the function  $f: [0,1] \to \mathbb{R}$  given by

$$f(x) = \begin{cases} 1, & \text{if } x = 1/3, \\ 2, & \text{if } x = 2/3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute the lower and upper integrals,  $\underline{\int}_{0}^{1} f(x) dx$  and  $\overline{\int}_{0}^{1} f(x) dx$ .

(b) Show that f is Riemann-integrable, and compute  $\int_0^1 f(x) dx$ .

8. Consider the function  $f: [1, \infty) \to \mathbb{R}$  given by

$$f(x) = \int_{1}^{\sqrt{x}} e^{t^2} dt$$

(a) What is f(1)?

(b) Show that f is differentiable. What is its derivative?

(c) What is f'(4)?