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NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

MATH 3150 **Real Analysis** **Fall 2011**

Final Exam

Put your name in the blanks above. Calculators are permitted. A single sheet of theorems and definitions is allowed. Provide explanations for all your answers. If needed, use the back of the page for additional space.

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1. Let $f\colon [2,3] \to \mathbb{R}$ be a function, continuous on $[2,3]$, and differentiable on $(2,3)$. Suppose that $f(2) = 6$ and $f(3) = 9$. Show that, for some point $x_0 \in (2,3)$, the tangent line to the graph of f at x_0 passes through the origin. Illustrate the result with a sketch.

2. (a) Give an example of a connected subset $A \subset \mathbb{R}$ such that its complement, $\mathbb{R} \setminus A$, is also connected.

(b) Give an example of a connected subset $A \subset \mathbb{R}$ such that its complement, $\mathbb{R} \setminus A$, *not* connected.

(c) Give an example of a family of open subsets $\{U_i\}_{i \in I}$ of $\mathbb{R}$ such that their intersection, $\bigcap_{i \in I} U_i$ is *not* open.

(d) Give an example of a family of closed subsets $\{C_i\}_{i \in I}$ of $\mathbb{R}$ such that their union, $\bigcup_{i \in I} C_i$ is *not* closed.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $G = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$ be its graph.

(a) Show that G is a closed subset of $\mathbb{R}^2$.

(b) Show that G is a connected subset of $\mathbb{R}^2$.

(c) Is G a compact subset of $\mathbb{R}^2$? Why, or why not?

4. Consider the following subset of the plane:

$$A = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{4} + y^2 \le 1 \right\}.$$

(a) Sketch the set A.

(b) Is A a closed subset of $\mathbb{R}^2$? Why, or why not?

(c) Is A a bounded subset of $\mathbb{R}^2$? Why, or why not?

(d) Is A a compact subset of $\mathbb{R}^2$? Why, or why not?

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two Lipschitz functions.

(a) Show that the composition of the two functions, $g \circ f\colon \mathbb{R} \to \mathbb{R}$, is also a Lipschitz function.

(b) Suppose now that both f and g are bounded functions. Show that the product of the two functions, $f \cdot g\colon \mathbb{R} \to \mathbb{R}$, is also a Lipschitz function.

(c) (Bonus) Give an example where $f \cdot g$ is *not* a Lipschitz function.

6. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{2\pi x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Show that f is continuous.

(b) Show that the restriction of f to the interval $[-1, 1]$ is uniformly continuous.

(c) Show that f is *not* differentiable at $x = 0$.

7. Consider the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x = 1/3, \\ 2, & \text{if } x = 2/3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute the lower and upper integrals, $\underline{\int_0^1} f(x)dx$ and $\overline{\int_0^1} f(x)dx$.

(b) Show that f is Riemann-integrable, and compute $\int_0^1 f(x)dx$.

8. Consider the function $f\colon [1,\infty) \to \mathbb{R}$ given by

$$f(x) = \int_1^{\sqrt{x}} e^{t^2}\, dt$$

(a) What is $f(1)$?

(b) Show that f is differentiable. What is its derivative?

(c) What is $f'(4)$?