Homework #2 Real Analysis

P48. 4. Pf of $\inf(A+B) = \inf A + \inf B$

(i) $\forall\ a \in A,\ b \in B$ $\inf A + \inf B \le a+b$ Hence $\inf A + \inf B$ is a lower bound of $A+B$.

(ii) $\forall\ \varepsilon > 0$ $\exists\ a \in A,\ b \in B$ s.t. $a < \inf A + \frac{\varepsilon}{2}$ $b < \inf B + \frac{\varepsilon}{2}$

So $a+b < \inf A + \inf B + \varepsilon$

Hence $\inf A + \inf B = \inf(A+B)$

P51-52 2. Pf. Every bounded real sequence has a convergent subsequence.

And $\forall\ n \in \mathbb{N}$ $e^{-1} \le x_n = e^{\sin(n)} \le e$

4. Pf $\{x_n\}$ is a Cauchy sequence.

Hence $\forall\ \varepsilon > 0,\ \exists N$ s.t. $\forall\ n, m \geqslant N$ $|x_n - x_m| < \frac{\varepsilon}{2}$

And by assumption $\exists\ M > \max\{\frac{2}{\varepsilon}, N\}$ s.t. $|x_M| < \frac{1}{\max\{\frac{2}{\varepsilon}, N\}} = \min\{\frac{\varepsilon}{2}, \frac{1}{N}\}$ $\le \frac{\varepsilon}{2}$

Take $m = \max\{N, M\}$ Then $\forall\ n \geqslant \max\{N, M\}$

$$|x_n| \le |x_n - x_m| + |x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $x_n \to 0$

5. Let $x_i = \begin{cases} 0, & i = 2k \quad k \in \mathbb{N} \\ \frac{1}{k-1}, & i = 2k+1 \quad k \in \mathbb{N} \end{cases}$ Let $n = 2k$ $m = 2l$ $k, l \in \mathbb{N}$

P98 9. Pf (i) Since $\forall\ n \in \mathbb{N}$, $y_n \le |x_n|$ $\limsup y_n \le \limsup |x_n|$.

($\limsup y_n = \inf\{\sup\{y_{n+1}, y_{n+2}, y_{n+3}, \cdots\} \mid n \in \mathbb{N}\}$)

(ii) No.

Let $x_n = (-1)^{n+1}$. Then $y_n = -1$. $|x_n| = 1$

$\limsup y_n = -1$. $\limsup |x_n| = 1$

(iii) $\liminf y_n \le \liminf |x_n|$.

...