Prof. Alexandru Suciu

MTH G321 **TOPOLOGY 3** **Spring 2009**

HOMEWORK 3

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1. Show that all the Whitehead products in the homotopy groups of an H-space vanish.

2. Let $\iota_n \in \pi_n(S^n)$ be the homotopy class of the identity map.

(a) Show that S^n is an H-space if and only if $[\iota_n, \iota_n] = 0$.

(b) Show that $[\iota_2, \iota_2] = 2\eta$, where η is the generator of $\pi_3(S^2)$ represented by the Hopf map.

3. Let $\alpha \in \pi_1(S^1 \vee S^2)$ and $\beta \in \pi_2(S^1 \vee S^2)$ be represented by the inclusion maps of the factors. Put

$$X = (S^1 \vee S^2) \cup_f D^3,$$

where $f\colon S^2 \to S^1 \vee S^2$ is a map representing $2\beta - \alpha \cdot \beta \in \pi_2(S^1 \vee S^2)$. Show that the inclusion map $i\colon S^1 \to X$ induces isomorphisms $i_\sharp\colon \pi_1(S^1) \xrightarrow{\simeq} \pi_1(X)$ and $i_*\colon H_n(S^1) \xrightarrow{\simeq} H_n(X)$ for all $n \geq 0$, though i is *not* a homotopy equivalence.

4. Let $f\colon X \to Y$ be a map between CW-complexes, such that the mapping cone C_f is contractible.

(a) Suppose both X and Y are simply-connected. Show that f is a homotopy equivalence.

(b) Give an example showing that the simply-connectivity assumption in part (a) cannot be dropped, in general.

5. Let $f\colon X \to Y$ be a map between connected CW-complexes. Show that f is a homotopy equivalence, provided either of the following two conditions holds.

(a) The induced homomorphism $f_\sharp\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism, and f admits a lift $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ to universal covers, such that the induced homomorphism, $\tilde{f}_*\colon H_n(\widetilde{X}) \to H_n(\widetilde{Y})$, is an isomorphism, for all $n \geq 0$.

(b) Both X and Y have dimension at most n, and the induced homomorphism, $f_\sharp\colon \pi_i(X) \to \pi_i(Y)$, is an isomorphism, for all $i \leq n$.