Cohomology jump loci of local systems

Botong Wang

Joint work with Nero Budur

University of Notre Dame

June 28 2013

Introduction

Given a topological space X, we can associate some homotopy invariants to it: the representation variety $\mathbf{R}(X, n)$ and cohomology jump loci $\mathcal{V}^i_k(X, n)$ $(n, i, k \in \mathbb{N})$. In this talk, we will discuss the deformation theoretic aspects of these invariants.

Definitions

Definition

Let X be a connected topological space of the homotopy type of a finite CW-complex. Fix a base point $x \in X$. The rank n **representation variety** *of X is defined to be*

$$\mathbf{R}(X, n) = Hom(\pi_1(X, x), Gl(n, \mathbb{C})).$$

Definitions

Definition

Let X be a connected topological space of the homotopy type of a finite CW-complex. Fix a base point $x \in X$. The rank n **representation variety** *of X is defined to be*

$$\mathbf{R}(X, n) = Hom(\pi_1(X, x), Gl(n, \mathbb{C})).$$

Definition

Under the same assumption, the rank n **cohomology jump loci** *of X is defined to be*

$$\mathcal{V}^i_k(X, n) = \{\rho \in \mathbf{R}(X, n) \mid \dim H^i(X, L_\rho) \geq k\}$$

where L_ρ is the local system associated to $\rho : \pi_1(X, x) \to Gl(n, \mathbb{C})$.

The theorem of Dimca-Papadima

The following is a partial generalization of earlier results of Green-Lazarsfeld, Arapura and Goldman-Millson.

Theorem (Dimca-Papadima)

The reduced analytic germ of $\mathcal{V}^i_k(X, n)$ at the trivial representation $\mathbb{1}$ only depends on the rational homotopy type of X.

The theorem of Dimca-Papadima

The following is a partial generalization of earlier results of Green-Lazarsfeld, Arapura and Goldman-Millson.

Theorem (Dimca-Papadima)

The reduced analytic germ of $\mathcal{V}^i_k(X, n)$ at the trivial representation $\mathbb{1}$ only depends on the rational homotopy type of X.

When X has some nice geometry, e.g., when X is a compact Kähler manifold, or quasi-Kähler manifold, the rational homotopy theory of X is particularly nice. In this case, one can obtain a nice description of the reduced analytic germs of $\mathcal{V}^i_k(X, n)$ at $\mathbb{1}$.

The theorem of Dimca-Papadima

The following is a partial generalization of earlier results of Green-Lazarsfeld, Arapura and Goldman-Millson.

Theorem (Dimca-Papadima)

The reduced analytic germ of $\mathcal{V}^i_k(X, n)$ at the trivial representation $\mathbb{1}$ only depends on the rational homotopy type of X.

When X has some nice geometry, e.g., when X is a compact Kähler manifold, or quasi-Kähler manifold, the rational homotopy theory of X is particularly nice. In this case, one can obtain a nice description of the reduced analytic germs of $\mathcal{V}^i_k(X, n)$ at $\mathbb{1}$.

In this talk, we want to generalize the above theorem in two directions. First, we want to have a result about the analytic germ at a general point in $\mathcal{V}^i_k(X, n)$. Second, we want to describe the whole analytic germ, not only the reduced part.

The precise definitions

Remark

$\mathbf{R}(X,n)$ *and* $\mathcal{V}^i_k(X,n)$ *are defined as sets in priori. However, they have natural (possibly non-reduced) scheme structures, which we will explain as follows.*

The precise definitions

Remark

$\mathbf{R}(X,n)$ and $\mathcal{V}^i_k(X,n)$ are defined as sets in priori. However, they have natural (possibly non-reduced) scheme structures, which we will explain as follows.

Given a finite set of generators of $\pi_1(X,x)$ and a finite set of relations, one can explicitly write down the equations defining $\mathbf{R}(X,n)$. Thus, $\mathbf{R}(X,n)$ has a scheme structure.

The precise definitions

Remark

$\mathbf{R}(X, n)$ and $\mathcal{V}^i_k(X, n)$ are defined as sets in priori. However, they have natural (possibly non-reduced) scheme structures, which we will explain as follows.

Given a finite set of generators of $\pi_1(X, x)$ and a finite set of relations, one can explicitly write down the equations defining $\mathbf{R}(X, n)$. Thus, $\mathbf{R}(X, n)$ has a scheme structure.

Over $X \times \mathbf{R}(X, n)$, there is a universal family of local systems, which we denote by $\mathcal{L}$. Denote the projections from $X \times \mathbf{R}(X, n)$ to the first and second factors by p_1 and p_2 respectively. One can represent $\mathbf{R}p_{2*}(\mathcal{L})$ by a bounded complex of free sheaves $(F^\bullet, d^\bullet)$, so that $H^i(X, L_\rho) \cong H^i((F^\bullet, d^\bullet)|_{\{\rho\}})$.

The precise definitions

Set-theoretically, $\mathcal{V}^i_k(X,n)$ the locus where

$$rank(F^i) - rank(d^{i-1}) - rank(d^i) \geq k.$$

Then precisely, $\mathcal{V}^i_k(X,n)$ as a subscheme of $\mathbf{R}(X,n)$ is defined by the determinantal ideal sheaf $I_{r_i-k+1}(d^{i-1} \oplus d^i)$, where r_i is the rank of F^i.

An overview

To obtain a statement about a general point of the representation variety, we will replace the notion of CDGA (commutative differential graded algebra) or DGLA (differential graded Lie algebra) by a DGLA pair, which consists of a DGLA and a module of that DGLA.

An overview

To obtain a statement about a general point of the representation variety, we will replace the notion of CDGA (commutative differential graded algebra) or DGLA (differential graded Lie algebra) by a DGLA pair, which consists of a DGLA and a module of that DGLA.

For results about the whole analytic germ, we will use the notion of cohomology jump ideals (determinantal ideals), instead of taking cohomology. This actually gives a more direct approach to describe the deformation theory of $\mathcal{V}^i_k(X, n)$.

Differential Graded Lie Algebra

Definition

A DGLA (differential graded Lie algebra) $(C^\bullet, d)$ *is a complex of* $\mathbb{C}$*-vector space* $(C^\bullet, d)$ *together with a Lie bracket* $[-,-] : C^i \times C^j \to C^{i+j}$*, satisfying the Leibnitz rule.*

Differential Graded Lie Algebra

Definition

A DGLA (differential graded Lie algebra) $(C^\bullet, d)$ is a complex of $\mathbb{C}$-vector space $(C^\bullet, d)$ together with a Lie bracket $[-,-] : C^i \times C^j \to C^{i+j}$, satisfying the Leibnitz rule.

Example

Any Lie algebra can be realized as a DGLA concentrating on degree zero.

Differential Graded Lie Algebra

Definition

A DGLA (differential graded Lie algebra) $(C^\bullet, d)$ *is a complex of* $\mathbb{C}$*-vector space* $(C^\bullet, d)$ *together with a Lie bracket* $[-,-] : C^i \times C^j \to C^{i+j}$*, satisfying the Leibnitz rule.*

Example

Any Lie algebra can be realized as a DGLA concentrating on degree zero.

Example

Let X be a connected smooth manifold, and let L be a local system on X. Then the global de Rham complex $\Omega^\bullet_{\mathrm{DR}}(End(L))$ of $End(L)$ is a DGLA. The Lie bracket is defined by $[f \otimes \theta, g \otimes \eta] = [f, g] \otimes \theta \wedge \eta$, where f and g are C^∞ global sections of $End(L)$, θ and η are differential forms on X.

Deformation theory attached to a DGLA

Definition

Given a DGLA $C^\bullet$, one can associate a deformation functor $Def(C^\bullet)$ from the category of Artinian local rings of finite type over $\mathbb{C}$ to the category of sets

$$A \mapsto \frac{\{\omega \in C^1 \otimes_{\mathbb{C}} A \,|\, d\omega + \frac{1}{2}[\omega, \omega] = 0\}}{\exp(C^0 \otimes_{\mathbb{C}} m)}$$

where m is the maximal ideal in A.

Deformation theory attached to a DGLA

Definition

Given a DGLA $C^\bullet$, one can associate a deformation functor $Def(C^\bullet)$ from the category of Artinian local rings of finite type over $\mathbb{C}$ to the category of sets

$$A \mapsto \frac{\{\omega \in C^1 \otimes_{\mathbb{C}} A \,|\, d\omega + \frac{1}{2}[\omega, \omega] = 0\}}{\exp(C^0 \otimes_{\mathbb{C}} m)}$$

where m is the maximal ideal in A.

We should consider the set to be the space of flat connections and $\exp(C^0 \otimes_{\mathbb{C}} m)$ acts on the set as “gauge transformation” (change of coordinates). The formula of the action is standard, but a little complicated.

Deformation theory attached to a DGLA

Let X be a connected smooth manifold with base point x, and let L be a local system on X. Let $\epsilon : \Omega^\bullet_{\mathrm{DR}}(End(L)) \to End(L)|_x$ be the restriction map. Denote the kernel of ϵ by $\Omega^\bullet_{\mathrm{DR}}(End(L))_\epsilon$. Then $\Omega^\bullet_{\mathrm{DR}}(End(L))_\epsilon$ is also a DGLA.

Deformation theory attached to a DGLA

Let X be a connected smooth manifold with base point x, and let L be a local system on X. Let $\epsilon : \Omega^{\bullet}_{\mathrm{DR}}(End(L)) \to End(L)|_x$ be the restriction map. Denote the kernel of ϵ by $\Omega^{\bullet}_{\mathrm{DR}}(End(L))_{\epsilon}$. Then $\Omega^{\bullet}_{\mathrm{DR}}(End(L))_{\epsilon}$ is also a DGLA.

Theorem (Goldman-Millson)

The coordinate ring of the analytic germ of $\mathbf{R}(X, n)$ at ρ represents the functor $Def(\Omega^{\bullet}_{DR}(End(L_{\rho}))_{\epsilon})$.

Deformation theory attached to a DGLA

Let X be a connected smooth manifold with base point x, and let L be a local system on X. Let $\epsilon : \Omega^\bullet_{\mathrm{DR}}(End(L)) \to End(L)|_x$ be the restriction map. Denote the kernel of ϵ by $\Omega^\bullet_{\mathrm{DR}}(End(L))_\epsilon$. Then $\Omega^\bullet_{\mathrm{DR}}(End(L))_\epsilon$ is also a DGLA.

Theorem (Goldman-Millson)

The coordinate ring of the analytic germ of $\mathbf{R}(X, n)$ *at* ρ *represents the functor* $Def(\Omega^\bullet_{DR}(End(L_\rho))_\epsilon)$.

Theorem (Deligne-Goldman-Millson-Schlessinger-Stasheff)

Suppose $f : C^\bullet \to D^\bullet$ *is 1-equivalent, i.e.,* f *induces isomorphisms on* H^0, H^1 *and monomorphism on* H^2. *Then the induced functor* $f_* : Def(C^\bullet) \to Def(D^\bullet)$ *is an equivalence of categories.*

Deformation theory attached to a DGLA

Remark

Suppose X is a compact Kähler manifold. Using the above two theorems and some formality result of Simpson, one can give a rather simple description of the analytic germ of the representation variety $\mathbf{R}(X, n)$ at a point corresponding to a semi-simple representation. In particular, such an analytic germ is quadratic.

Deformation theory attached to a DGLA

Remark

Suppose X is a compact Kähler manifold. Using the above two theorems and some formality result of Simpson, one can give a rather simple description of the analytic germ of the representation variety $\mathbf{R}(X, n)$ at a point corresponding to a semi-simple representation. In particular, such an analytic germ is quadratic.

We want to find a similar theory, which works more generally for cohomology jump loci.

DGLA pairs

Definition

Let $(C^\bullet, d_C)$ be a DGLA. A module of $C^\bullet$ is a complex of $\mathbb{C}$-vector spaces $(M^\bullet, d_M)$ together with a bilinear multiplication map $\cdot : C^i \times M^j \to M^{i+j}$ satisfying the following identities

$$d_M(\alpha \cdot s) = d_C\alpha \cdot s + (-1)^{\deg(\alpha)}\alpha \cdot d_M s$$

and

$$[\alpha, \beta] \cdot s = \alpha \cdot (\beta \cdot s) - (-1)^{\deg(\alpha)\deg(\beta)}\beta \cdot (\alpha \cdot s).$$

DGLA pairs

Definition

Let $(C^\bullet, d_C)$ be a DGLA. A module of $C^\bullet$ is a complex of $\mathbb{C}$-vector spaces $(M^\bullet, d_M)$ together with a bilinear multiplication map $\cdot : C^i \times M^j \to M^{i+j}$ satisfying the following identities

$$d_M(\alpha \cdot s) = d_C\alpha \cdot s + (-1)^{\deg(\alpha)}\alpha \cdot d_M s$$

and

$$[\alpha, \beta] \cdot s = \alpha \cdot (\beta \cdot s) - (-1)^{\deg(\alpha)\deg(\beta)}\beta \cdot (\alpha \cdot s).$$

Definition

A DGLA pair consists of a DGLA $C^\bullet$ and a $C^\bullet$-module $M^\bullet$. We write such a DGLA pair by $(C^\bullet, M^\bullet)$.

DGLA pairs

Remark

Consider $(M^\bullet, d_M)$ *as a complex of vector spaces. Then* $End(M^\bullet)$ *has a natural DGLA structure. In fact,*

$$End^j(M^\bullet) = \bigoplus_i Hom(M^i, M^{i+j})$$

and

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \in End(M^\bullet).$$

The multiplication $\cdot\ : C^\bullet \times M^\bullet \to M^\bullet$ *is equivalent to a linear map*

$$\cdot\ : C^\bullet \to End(M^\bullet).$$

The condition on the multiplication is equivalent to the above morphism being a morphism of DGLA. In this sense, it is considered earlier by M. Manetti.

DGLA pairs

Example

Under the same notations as before, $(\Omega^{\bullet}_{\mathrm{DR}}(End(L)), \Omega^{\bullet}_{\mathrm{DR}}(L))$ is a DGLA pair.

DGLA pairs

Example

Under the same notations as before, $(\Omega^\bullet_{\mathrm{DR}}(End(L)), \Omega^\bullet_{\mathrm{DR}}(L))$ is a DGLA pair.

Example

Let X be a connected complex manifold, and let E be a holomorphic vector bundle. We use $\Omega^\bullet_{\mathrm{Dol}}$ to denote the Dolbeault complex of a holomorphic vector bundle. Then $(\Omega^\bullet_{\mathrm{Dol}}(End(E)), \Omega^\bullet_{\mathrm{Dol}}(E \otimes_{\mathcal{O}_X} \Omega^p_X))$ is a DGLA pair.

DGLA pairs

Example

Under the same notations as before, $(\Omega^{\bullet}_{\mathrm{DR}}(End(L)), \Omega^{\bullet}_{\mathrm{DR}}(L))$ is a DGLA pair.

Example

Let X be a connected complex manifold, and let E be a holomorphic vector bundle. We use $\Omega^{\bullet}_{\mathrm{Dol}}$ to denote the Dolbeault complex of a holomorphic vector bundle. Then $(\Omega^{\bullet}_{\mathrm{Dol}}(End(E)), \Omega^{\bullet}_{\mathrm{Dol}}(E \otimes_{\mathcal{O}_X} \Omega^p_X))$ is a DGLA pair.

Using this DGLA pair, we can study the H^{pq} cohomology jump loci in the moduli space of stable holomorphic vector bundles on X.

DGLA pairs

Definition

A morphism $f : (C^\bullet, M^\bullet) \to (D^\bullet, N^\bullet)$ *of DGLA pairs consists of a morphism of DGLA* $f_1 : C^\bullet \to D^\bullet$ *and a morphism of complexes* $f_2 : M^\bullet \to N^\bullet$ *which is compatible with the multiplication map.*

DGLA pairs

Definition

A morphism $f : (C^\bullet, M^\bullet) \to (D^\bullet, N^\bullet)$ *of DGLA pairs consists of a morphism of DGLA* $f_1 : C^\bullet \to D^\bullet$ *and a morphism of complexes* $f_2 : M^\bullet \to N^\bullet$ *which is compatible with the multiplication map.*

Definition

A morphism of complexes is called q-equivalent, if it induces isomorphism on i*-th cohomology for* $i \leq q$ *and monomorphism on* $(q+1)$*-th cohomology. A morphism* $f : (C^\bullet, M^\bullet) \to (D^\bullet, N^\bullet)$ *of DGLA pairs is called q-equivalent, if* $f_1 : C^\bullet \to D^\bullet$ *is 1-equivalent and* $f_2 : M^\bullet \to N^\bullet$ *is q-equivalent. Two DGLA pairs are called of the same q-homotopy type, if they can be connected by a zigzag of q-equivalent morphisms.*

The main goal

Simiar to the theorems of D-G-M-S-S and G-M, we will define a deformation functor $Def^i_k(C^\bullet, M^\bullet)$ associated to any DGLA pair $(C^\bullet, M^\bullet)$. Moreover, we want the parallel theorems to hold.

The main goal

Simiar to the theorems of D-G-M-S-S and G-M, we will define a deformation functor $Def_k^i(C^\bullet, M^\bullet)$ associated to any DGLA pair $(C^\bullet, M^\bullet)$. Moreover, we want the parallel theorems to hold.

Theorem

Suppose $\rho \in \mathcal{V}_k^i(X, n)$. Then the analytic germ of $\mathcal{V}_k^i(X, n)$ at ρ represents the functor $Def_k^i(\Omega^\bullet_{DR}(End(L_\rho))_\epsilon, \Omega^\bullet_{DR}(L_\rho))$.

The main goal

Simiar to the theorems of D-G-M-S-S and G-M, we will define a deformation functor $Def_k^i(C^\bullet, M^\bullet)$ associated to any DGLA pair $(C^\bullet, M^\bullet)$. Moreover, we want the parallel theorems to hold.

Theorem

Suppose $\rho \in \mathcal{V}_k^i(X, n)$. Then the analytic germ of $\mathcal{V}_k^i(X, n)$ at ρ represents the functor $Def_k^i(\Omega^\bullet_{DR}(End(L_\rho))_\epsilon, \Omega^\bullet_{DR}(L_\rho))$.

Theorem

The deformation functor $Def_k^i(C^\bullet, M^\bullet)$ only depends on the i-th homotopy type of the DGLA pair $(C^\bullet, M^\bullet)$.

Formal DGLA pairs

Definition

A DGLA pair $(C^\bullet, M^\bullet)$ *is called formal, if it is of the same homotopy type as* $(H^\bullet(C^\bullet), H^\bullet(M^\bullet))$.

Formal DGLA pairs

Definition

A DGLA pair $(C^\bullet, M^\bullet)$ is called formal, if it is of the same homotopy type as $(H^\bullet(C^\bullet), H^\bullet(M^\bullet))$.

Proposition (Simpson)

If X is a compact Kähler manifold, and if L is a local system whose monodromy representation is semi-simple, then $(\Omega^\bullet_{DR}(End(L)), \Omega^\bullet_{DR}(L))$ is formal.

Formal DGLA pairs

Corollary

Let X be a compact Kähler manifold and let L be a local system whose monodromy representation is semi-simple. Suppose $L \in \mathcal{V}^i_k(X, n)$. We define

$$\mathcal{R}^i_k(X, L) = \{\xi \in H^1(X, End(L)) \,|\, \xi \wedge \xi = 0, \dim H^i(H^\bullet(L), \wedge\xi) \geq k\}$$

Then there is an isomorphism between analytic germs

$$\mathcal{V}^i_k(X, n)_{(L)} \cong \mathcal{R}^i_k(X, L)_{(0)}$$

Cohomology jump ideal

To define the deformation functor Def^i_k, we need first the notion of cohomology jump ideals, which is similar to fitting ideals.

Cohomology jump ideal

To define the deformation functor Def_k^i, we need first the notion of cohomology jump ideals, which is similar to fitting ideals.

Definition-Proposition

Let R be a noetherian ring, and let $K^\bullet$ be a bounded above complex of R-modules. Suppose $H^i(K^\bullet)$ is a finitely generated R-module for every $i \in \mathbb{Z}$. Then there exists a bounded above complex of finitely generated free R-modules $(F^\bullet, d^\bullet)$, which is quasi-isomorphic to $K^\bullet$. The cohomology jump ideal $J_k^i(K^\bullet)$ is defined to be the determinantal ideal $I_{l_i-k+1}(d^{i-1} \oplus d^i)$, where l_1 is the rank of F^i. This definition is independent of the choice of $F^\bullet$.

Cohomology jump ideal

To define the deformation functor Def_k^i, we need first the notion of cohomology jump ideals, which is similar to fitting ideals.

Definition-Proposition

Let R be a noetherian ring, and let $K^\bullet$ be a bounded above complex of R-modules. Suppose $H^i(K^\bullet)$ is a finitely generated R-module for every $i \in \mathbb{Z}$. Then there exists a bounded above complex of finitely generated free R-modules $(F^\bullet, d^\bullet)$, which is quasi-isomorphic to $K^\bullet$. The cohomology jump ideal $J_k^i(K^\bullet)$ is defined to be the determinantal ideal $I_{l_i-k+1}(d^{i-1} \oplus d^i)$, where l_1 is the rank of F^i. This definition is independent of the choice of $F^\bullet$.

Remark

When R is a field, $\dim H^i(K^\bullet) \geq k$ if and only if $J_k^i(K^\bullet) = 0$. $J_k^i(K^\bullet) = 0$ is the analog for $\mathrm{rank}_R H^i(K^\bullet) \geq k$, which does not make sense when R is an Artinian local algebra.

Definition of the deformation functor Def_k^i

Let $(C^\bullet, M^\bullet)$ be a DGLA pair, and suppose $H^q(M^\bullet)$ is a finite dimensional $\mathbb{C}$-vector space for every $q \in \mathbb{Z}$.

Definition of the deformation functor Def_k^i

Let $(C^\bullet, M^\bullet)$ be a DGLA pair, and suppose $H^q(M^\bullet)$ is a finite dimensional $\mathbb{C}$-vector space for every $q \in \mathbb{Z}$. Recall that $Def(C^\bullet)$ is the functor from the category of Artinian local $\mathbb{C}$-algebras to the category of sets defined by

$$Def(C^\bullet)(A) = \frac{\{\omega \in C^1 \otimes_{\mathbb{C}} A \mid d\omega + \frac{1}{2}[\omega, \omega] = 0\}}{\exp(C^0 \otimes_{\mathbb{C}} m)}.$$

Definition of the deformation functor Def^i_k

Let $(C^\bullet, M^\bullet)$ be a DGLA pair, and suppose $H^q(M^\bullet)$ is a finite dimensional $\mathbb{C}$-vector space for every $q \in \mathbb{Z}$. Recall that $Def(C^\bullet)$ is the functor from the category of Artinian local $\mathbb{C}$-algebras to the category of sets defined by

$$Def(C^\bullet)(A) = \frac{\{\omega \in C^1 \otimes_{\mathbb{C}} A \,|\, d\omega + \frac{1}{2}[\omega, \omega] = 0\}}{\exp(C^0 \otimes_{\mathbb{C}} m)}.$$

Given any $\{\omega \in C^1 \otimes_{\mathbb{C}} A \,|\, d\omega + \frac{1}{2}[\omega, \omega] = 0\}$, there is a complex $(M^\bullet \otimes_{\mathbb{C}} A, d_\omega)$, where $d_\omega = d \otimes id_A + \omega$. The condition $d\omega + \frac{1}{2}[\omega, \omega] = 0$ implies $d_\omega^2 = 0$.

Definition of the deformation functor Def_k^i

Proposition

The action of $\exp(C^0 \otimes_{\mathbb{C}} m)$ *on "the space of flat connections"* $\left\{\omega \in C^1 \otimes_{\mathbb{C}} A \,|\, d\omega + \frac{1}{2}[\omega,\omega] = 0\right\}$ *preserves the subset*

$$\left\{\omega \in C^1 \otimes_{\mathbb{C}} A \,\middle|\, d\omega + \frac{1}{2}[\omega,\omega] = 0, J_k^i(M^\bullet \otimes_{\mathbb{C}} A, d_\omega) = 0\right\}.$$

Definition of the deformation functor Def_k^i

Proposition

The action of $\exp(C^0 \otimes_{\mathbb{C}} m)$ *on "the space of flat connections"* $\left\{\omega \in C^1 \otimes_{\mathbb{C}} A \,|\, d\omega + \frac{1}{2}[\omega, \omega] = 0\right\}$ *preserves the subset*

$$\left\{\omega \in C^1 \otimes_{\mathbb{C}} A \,|\, d\omega + \frac{1}{2}[\omega, \omega] = 0, J_k^i(M^\bullet \otimes_{\mathbb{C}} A, d_\omega) = 0\right\}.$$

Thus we can define:

Definition

We define $Def_k^i(C^\bullet, M^\bullet)$ *to be the subfunctor of* $Def(C)$ *given by*

$$A \mapsto \frac{\left\{\omega \in C^1 \otimes_{\mathbb{C}} A \,|\, d\omega + \frac{1}{2}[\omega, \omega] = 0, J_k^i(M^\bullet \otimes_{\mathbb{C}} A, d_\omega) = 0\right\}}{\exp(C^0 \otimes_{\mathbb{C}} m)}$$

Some remarks

Remark

The DGLA pair associated to the origin of $\mathbf{R}(X, n)$ *is*

$$(\Omega^{\bullet}_{DR}(X) \otimes_{\mathbb{C}} End(\mathbb{C}^n), \Omega^{\bullet}_{DR}(X) \otimes_{\mathbb{C}} \mathbb{C}^n).$$

Thus we recover the theorem of Dimca-Papadima that the analytic germ of the cohomology jump loci at origin only depends on the rational homotopy type of X*.*

Some remarks

Remark

The DGLA pair associated to the origin of $\mathbf{R}(X, n)$ *is*

$$(\Omega^{\bullet}_{DR}(X) \otimes_{\mathbb{C}} End(\mathbb{C}^n), \Omega^{\bullet}_{DR}(X) \otimes_{\mathbb{C}} \mathbb{C}^n).$$

Thus we recover the theorem of Dimca-Papadima that the analytic germ of the cohomology jump loci at origin only depends on the rational homotopy type of X.

Remark

Using DGLA pairs, we can not only study the cohomology jump loci in the representation variety, we can also study the cohomology jump loci in the moduli space of stable vector bundles, in the moduli space of irreducible local systems, or in the moduli space of Higgs bundles.

Some remarks

Remark

Using DGLA pairs, we can study the relative cohomology jump loci. For example, fixing any local system V, we can define the relative cohomology jump loci

$$\mathcal{V}^i_k(X, n; V) = \{\rho \in \mathbf{R}(X, n) \mid \dim H^i(X, L_\rho \otimes_{\mathbb{C}} V) \geq k\}.$$

Now, the DGLA pair controlling the deformation problem is $(\Omega^\bullet_{DR}(End(L_\rho))_\epsilon, \Omega^\bullet_{DR}(L_\rho \otimes_{\mathbb{C}} V))$

Some remarks

Remark

Using DGLA pairs, we can study the relative cohomology jump loci. For example, fixing any local system V, we can define the relative cohomology jump loci

$$\mathcal{V}^i_k(X, n; V) = \{\rho \in \mathbf{R}(X, n) \mid \dim H^i(X, L_\rho \otimes_{\mathbb{C}} V) \geq k\}.$$

Now, the DGLA pair controlling the deformation problem is $(\Omega^\bullet_{DR}(End(L_\rho))_\epsilon, \Omega^\bullet_{DR}(L_\rho \otimes_{\mathbb{C}} V))$

Remark

Instead of $Gl(n, \mathbb{C})$ representations, one can also consider the representation variety of any complex reductive algebraic group. The whole theory will work the same.

Mulțumesc!