Cohomology jump loci of quasi-projective varieties

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joint work with Nero Budur

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Motivation

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In this talk, we will give some constraints of the homotopy type of smooth complex quasi-projective varieties.

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We denote the space of rank one local system on X by $\mathcal{M}_B(X)$. $\mathcal{M}_B(X)$ is isomorphic to the direct product of $(\mathbb{C}^*)^{b_1(X)}$ and a finite abelian group, and hence it is an abelian algebraic group.

Non-abelian Hodge theory and cohomology jump loci

The non-abelian Hodge theory due to Simpson says "the space of local systems is equal to the space of Higgs bundles". In the rank one case, it says when X is a compact Kähler manifold,

$$\mathcal{M}_B(X) \overset{\text{by definition}}{=\!=} Hom(\pi_1(X), \mathbb{C}^*) \cong Pic^\tau(X) \times H^0(X, \Omega_X^1).$$

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However, we will not use this at all in this talk. Instead, we need to introduce our main player, the **cohomology jump loci** in $\mathcal{M}_B(X)$,

$$\Sigma^i_k(X) \overset{\text{def}}{=} \{L \in \mathcal{M}_B(X) \mid \dim H^i(X, L) \geq k\}.$$

They are closed subvarieties defined over $\mathbb{Q}$.

The result of Green-Lazarsfeld, Arapura and Simpson

The cohomology jump locus is a general notion defined for any connected topological space of the homotopy type of a finite CW-complex. It is a homotopy invariant. *In general, it can be any subvariety of a torus.*

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Theorem (Green-Lazarsfeld, Arapura)

If X is a compact Kähler manifold, then $\Sigma^i_k(X) \subset \mathcal{M}_B(X)$ is finite union of translates of subtori, for any $i, k \in \mathbb{N}$.

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These two theorems give nontrivial constraints on the possible homotopy type of compact Kähler manifolds and smooth complex projective varieties.

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Examples

Example (A trivial example)

Let X be a compact Riemann surface of genus $g \geq 1$. Since $H_1(X, \mathbb{Z}) = \mathbb{Z}^{2g}$, $\mathcal{M}_B(X) = (\mathbb{C}^*)^{2g}$.

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$$H^0(X, \underline{\mathbb{C}}_X) = \mathbb{C},\ H^1(X, \underline{\mathbb{C}}_X) = \mathbb{C}^{2g},\ H^2(X, \underline{\mathbb{C}}_X) = \mathbb{C}.$$

Suppose L is a rank one local system on X which is not isomorphic to the trivial local system $\underline{\mathbb{C}}_X$. Then

$$H^0(X, L) = 0,\ H^1(X, L) = \mathbb{C}^{2g-2},\ H^2(X, L) = 0.$$

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Therefore, all the cohomology jump loci are either the whole space $\mathcal{M}_B(X)$, or the single point at the trivial local system $\{\underline{\mathbb{C}}_X\}$, or empty.

Examples

Example (nontrivial)

Let $\sigma = \{\sigma_{ij}\}$ be an invertible $n \times n$ matrix with entries in $\mathbb{Z}$. Suppose none of the eigenvalues of σ is a root of unity. Define an action of $\mathbb{Z}$ on $(\mathbb{C}^*)^n \times \mathbb{C}$ by $l \cdot (\eta, \lambda) = (\sigma^l \cdot \eta, \lambda + l)$, where $l \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, $\eta = \{\eta_k\} \in (\mathbb{C}^*)^n$ and $\sigma^l \cdot \eta = \{\prod_j \eta_j^{\sigma_{ij}}\}_{1 \le i \le n}$.
Now, denote by X the quotient of $(\mathbb{C}^*)^n \times \mathbb{C}$ by $\mathbb{Z}$ under the above action. Since $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$, one can easily see that X is a $(\mathbb{C}^*)^n$ fiber bundle over $\mathbb{C}^*$.

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Now, denote by X the quotient of $(\mathbb{C}^*)^n \times \mathbb{C}$ by $\mathbb{Z}$ under the above action. Since $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$, one can easily see that X is a $(\mathbb{C}^*)^n$ fiber bundle over $\mathbb{C}^*$.
First, we compute $\pi_1(X)$. Since X is a quotient of $(\mathbb{C}^*)^n \times \mathbb{C}$ by $\mathbb{Z}$, $\pi_1(X) \cong \mathbb{Z}^n \rtimes_\sigma \mathbb{Z}$. The condition that none of the eigenvalues of σ is a root of unity implies that the normal subgroup $\mathbb{Z}^n$ is (almost) equal to the commutator subgroup of $\pi_1(X)$. Hence $H_1(X, \mathbb{Z}) \cong \mathbb{Z}$, and $\mathcal{M}_B(X) \cong Hom(H_1(X), \mathbb{C}^*) \cong \mathbb{C}^*$.

Examples

Example (continued)

Next, we compute the cohomology jump loci $\Sigma^1_1(X)$. For this example, the best way to compute the cohomology of a local system is to use group cohomology.

Recall that a rank one local system L on X is equivalent to a $\mathbb{C}^*$ representation of $\pi_1(X)$, which we denote by ρ_L. $\rho_L : \pi_1(X) \to \mathbb{C}^*$ puts a $\pi_1(X)$-module structure on $\mathbb{C}$ by $\alpha \cdot t = \rho_L(\alpha)t$, where $\alpha \in \pi_1(X)$ and $t \in \mathbb{C}$. We write $\mathbb{C}_{\rho_L}$ to emphasize the $\pi_1(X)$-module structure on $\mathbb{C}$.

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Since the universal cover of X is contractible,

$$H^i(X, L) \cong H^i(\pi_1(X), \mathbb{C}_{\rho_L}).$$

Now, using group cohomology, it is not hard to see that

$$\Sigma_1^1(X) = \{\text{eigenvalues of } \sigma\} \cup \{1\} \subset \mathbb{C}^* = \mathcal{M}_B(X).$$

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Conclusion

Since $\{$*eigenvalues of* $\sigma\}$ *consists of points that are not torsion, by the theorem of Budur-W.,* X *is not homotopy equivalent to any quasi-projective variety.*

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Remark

Even when the universal cover of a topological space is not contractible, the first cohomology of a local system can always be computed via group cohomology, and hence only depends on the fundamental group. Thus, both the theorems of Dimca-Papadima and Budur-W. put nontrivial constraints on the possible fundamental groups of a quasi-projective varieties. For example, $\mathbb{Z}^n \rtimes_\sigma \mathbb{Z}$ is not isomorphic to the fundamental group of any quasi-projective variety.

Proof of Theorem of Budur-W.

The idea behind the results of Green-Lazarsfeld, Arapura and Dimca-Papadima can be summarized into one sentence: *"the obstruction of deforming elements in the cohomology group is linear."*

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However, the proof of the theorem of Budur-W. is arithmetic. We will first work with quasi-projective varieties that are defined over $\bar{\mathbb{Q}}$.

Let X be a smooth complex quasi-projective variety, and let $\bar{X}$ be a good compactification. In other words, X is a smooth projective variety and $D = \bar{X} - X$ is a simple normal crossing divisor. Denote the number of irreducible components of D by n.

Proof of Theorem of Budur-W.

The following diagram plays an essential role in our proof.

$$
\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & 0 & & \mathbb{Z}^n & = & \mathbb{Z}^n \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}_{DR}(\bar{X}) & \longrightarrow & \mathcal{M}_{DR}(\bar{X}/D) & \xrightarrow{res} & \mathbb{C}^n \\
 & & \downarrow RH & & \downarrow RH & & \downarrow \exp \\
0 & \longrightarrow & \mathcal{M}_B(\bar{X}) & \longrightarrow & \mathcal{M}_B(X) & \xrightarrow{ev} & (\mathbb{C}^*)^n \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
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$\mathcal{M}_{DR}(\bar{X})$ is the moduli space of rank one flat bundles on $\bar{X}$.
$\mathcal{M}_{DR}(\bar{X}/D)$ is the moduli space of rank one flat bundles on $\bar{X}$ with possible log poles along D.
res is the taking residue map.
ev is taking monodromy around each D_i.
RH are the maps of Riemann-Hilbert correspondence.
ex is the composition of multiplication by $2\pi i$ and then taking exponential.

Proof of Theorem of Budur-W.

$\Sigma^i_k(X) \subset \mathcal{M}_B(X)$ is defined over $\mathbb{Q}$. Using the hypercohomology of logarithmic de Rham complexes, one can define similarly cohomology jump loci $\Sigma^i_k(\bar{X}/D)$ in $\mathcal{M}_{DR}(\bar{X}/D)$. Away from some small bad locus, $RH(\Sigma^i_k(\bar{X}/D)) = \Sigma^i_k(X)$.

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Proposition

Suppose $V \subset \mathcal{M}_{DR}(\bar{X}/D)$ and $W \subset \mathcal{M}_B(X)$ are closed irreducible subvarieties defined over $\bar{\mathbb{Q}}$. Moreover, suppose $RH(V) = W$. Then W consists of a torsion point.

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The proposition essentially follows from the next two theorems.

Proof of Theorem of Budur-W.

Theorem (Simpson)

Suppose $A \subset \Sigma^i_k(\bar{X})$ *and* $B \subset \mathcal{M}_B(\bar{X})$ *are closed irreducible subvarieties defined over* $\bar{\mathbb{Q}}$*. Moreover, suppose* $RH(A) = B$*. Then* B *consists of a torsion point.*

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Theorem (Gelfond-Schneider)

Suppose $C \subset \mathbb{C}^n$ and $D \subset (\mathbb{C}^*)^n$ are both defined over $\bar{\mathbb{Q}}$, and suppose $\exp(C) = D$. Then D consists of a torsion point.

Proof of Theorem of Budur-W.

Now, it follows from the proposition that any irreducible component of $\Sigma^i_k(X)$ consists of a torsion point. By taking a finite cover of X, one can move the point to origin. Then the theorem of Budur-W. follows from the theorem of Dimca-Papadima. Thus we have proved that when X is defined over $\bar{\mathbb{Q}}$, $\Sigma^i_k(X)$ is finite union of torsion translates of subtori.

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In general, one can prove that for any smooth complex quasi-projective variety X, there exists another smooth complex quasi-projective variety X' defined over $\bar{\mathbb{Q}}$ such that X' is homeomorphic to X. Since our statement is topological, it is true for any smooth complex quasi-projective variety.

Some further questions

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If X is a compact Kähler manifold or a quasi-Kähler manifold, does every irreducible component of $\Sigma^i_k(X)$ contain a torsion point?

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Question

How about higher rank analog?

Some further questions

When X is defined over $\bar{\mathbb{Q}}$, it seems plausible to replace torsion point by a $\mathbb{Q}$-variation of Hodge structure whose Gauss-Manin connection is defined over $\bar{\mathbb{Q}}$.

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Being more optimistic, Arapura made the following conjecture, which refines a conjecture of Simpson.

Conjecture

When X is a smooth projective variety, every nonempty $\Sigma^i_k(X)$ contains a motivic point (a point of geometric origin).

Mulțumesc!