Enumerative geometry of hyperplane arrangements

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Outline

- Introduction
- Warm up examples
- Multivariate Tutte Polynomial
- Characteristic Numbers of generic arrangements
- Counting Cones of generic arrangements

Classical enumerative geometry

"Counting some algebraic varieties that satisfy certain geometric conditions."

Typical problems:

• How many conic sections are tangent to five given lines in the projective plane?

• How many lines in $\mathbb{R}^3$ pass through 4 general lines?

Note: Usually the varieties in these problems do not have much more structure than their dimension and degree.

Arrangement setting

Choose a matroid or geometric lattice L with rank $r+1$.

$\mathcal{M}(L)$="the set of hyp arr's in $\mathbb{P}^r$ with lattice L"

Main question: What is the degree N_L of $\mathcal{M}(L)$?

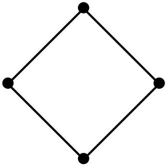
Classical Enumerative Geometry view:

• Let $D = \dim \mathcal{M}(L)$

• Fix D general position points in $\mathbb{P}^r$.

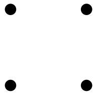
• How many arrangements N_L with intersection lattice $\cong L$ contain these D points?

Easy example

Let $L =$ 

Then $r = 2$ and $D = 4$ but view this in $\mathbb{P}^2$.
Question: How many different pairs of lines in $\mathbb{P}^2$ contain 4 points?

Answer: $N_L = \binom{4}{2}/2! = 3$



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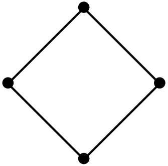
Let $L =$ 

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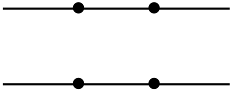

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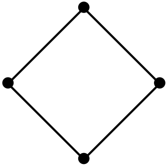
Let $L =$ 

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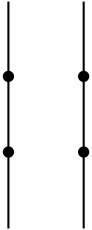

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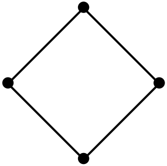
Let $L =$ 

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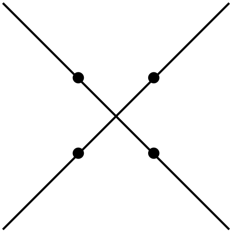


Generic Arrangements

An arrangement $\mathcal{G}_{n,k} = \{H_1, \ldots, H_k\}$ in $\mathbb{P}^n$ is generic if the intersection of any $n+1$ hyperplanes

$$H_{i_1} \cap \cdots \cap H_{i_{n+1}} = \vec{0}$$

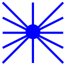
$\dim \mathcal{M}(G_{n,k}) = nk$

Theorem (Carlini)

The number of generic arrangements of size k in $\mathbb{P}^n$ through nk points is

$$N_{\mathcal{G}_{n,k}} = \frac{1}{k!}\binom{kn}{n}\binom{(k-1)n}{n}\cdots\binom{n}{n} = \frac{(kn)!}{k!(n!)^k}.$$

This came up when studying the Chow variety of zero dimensional degree k cycles in $\mathbb{P}^n$.

Star arrangements in $\mathbb{P}^2$: 

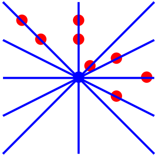
A star arrangement $S_k = \{H_1, \ldots, H_k\}$ in $\mathbb{P}^2$ has

$$\bigcap_{l=1}^{k} H_i = pt.$$

$\dim \mathcal{M}(S_k) = k + 2$

Proposition: The number of star arrangements S_k that contain $k + 2$ points is

$$N_{S_k} = \binom{k+2}{2, 2, k-2} / 2 = 3\binom{k+2}{4}$$



Multivariate Tutte polynomial

The multivariate Tutte polynomial of an arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$ is

$$Z_{\mathcal{A}}(q, v_1, \ldots, v_k) = \sum_{\mathcal{B} \subseteq \mathcal{A}} q^{-rk(\mathcal{B})} \prod_{H_i \in \mathcal{B}} v_i$$

$\mathcal{G}_{2,k}$ – a generic arrangement in $\mathbb{P}^2$

Fact: $N_{\mathcal{G}_{2,k}} = Z_{\mathcal{G}_{2,k}}(1, 0, 2, 4, \ldots, 2(k-1)) = (2k-1)!!$

Characteristic numbers

For an arrangement $\mathcal{A}$ in $\mathbb{P}^n$ and integers p, ℓ such that $p + \ell = \dim \mathcal{M}(\mathcal{A})$ the characteristic numbers are

$N_{\mathcal{A}}(p, \ell) =$ the number of arrangements combinatorially

equivalent to $\mathcal{A}$ that contain p points and are tangent to ℓ lines

- $N_{\mathcal{A}} = N_{\mathcal{A}}(\dim \mathcal{M}(\mathcal{A}), 0)$
- $N_{\mathcal{A}}(p, \ell)$ are in general very difficult to compute
- Usually if you can compute all the characteristic numbers for your object then you can compute all enumerative problems with that object.
- To compute this we will need the class of a curve is the number of lines passing through a given general point and tangent to the curve at a simple point. For example, the class of a smooth curve of degree d is $d(d-1)$.

Characteristic polynomial

Adapting a Fulton-MacPherson theorem to line arrangements in $\mathbb{P}^2$ we get:

Theorem

The number of line arrangements with intersection lattice isomorphic to $L_{\mathcal{A}}$ through p points and tangent to $D-p$ smooth curves of degrees $n_1, \ldots, n_{D-p}$ and classes $m_1, \ldots, m_{D-p}$ in general position is

- *write down*

$$\mathcal{C} = \mu^p \prod_{i=1}^{D-p} (m_i\mu + n_i\nu)$$

- *expand the polynomial $\mathcal{C}$*
- *plug in the characteristic numbers for each term in the expansion $N_{\mathcal{A}}(k, D-k) = \mu^k \nu^{D-k}$*
- *sum all terms.*

3 and 4 generic lines in $\mathbb{P}^2$

Theorem (Paul, Traves, W.)

p	0	1	2	3	4	5	6
$N_{\mathcal{G}_{2,3}}(p, 6-p)$	15	30	48	57	48	30	15

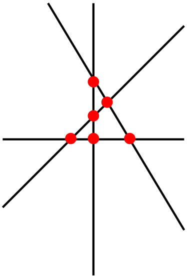
Theorem (Paul, Traves, W.)

p	0	1	2	3	4	5	6	7	8
$N_{\mathcal{G}_{2,4}}(p, 8-p)$	16695	17955	13185	8190	4410	2070	855	315	105

- Do each example separately.
- Examine the Chow ring of $A = A[(\mathbb{P}^{2*})^k \times (\mathbb{P}^2)^s]$ where $s = |L(\mathcal{A})_2|$ =the number of intersection points of lines in $\mathcal{A}$.
- $A = A[(\mathbb{P}^{2*})^k \times (\mathbb{P}^2)^s] \cong \frac{\mathbb{Z}[x_1,\ldots,x_k,y_1,\ldots,y_s]}{(x_1^3,\ldots,x_k^3,y_1^3,\ldots,y_s^3)}$
- Form a class $[\mathcal{M}(\mathcal{A})] \in A$ that represents the moduli space and the tangency conditions.
- Expand this class in A.
- The coefficient of this class is $N_{\mathcal{A}}(p, \ell)$
- WARNING: Many of these cases have excess intersection and multiplicities that must be accounted for.

$N_{\mathcal{G}_{2,4}}(0, 8)$

The projective dual of $\mathcal{G}_{2,4}$ is the braid arrangement A_3



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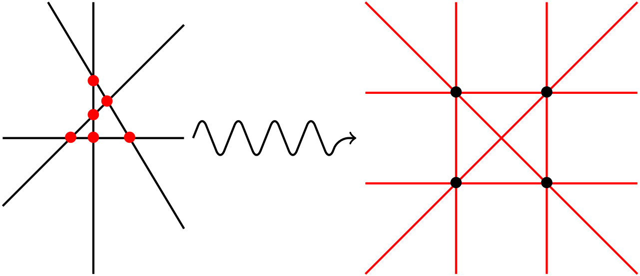

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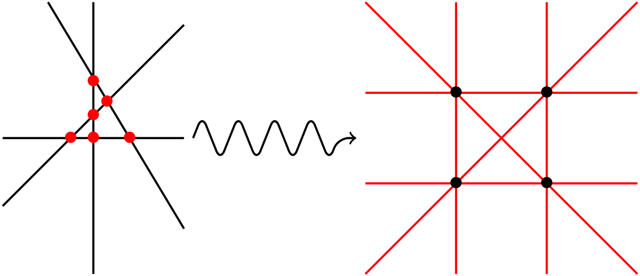

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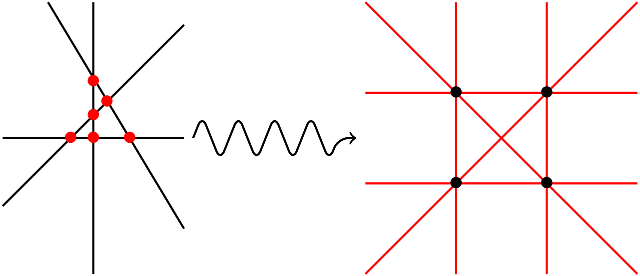

The dual of the 8 line conditions for $\mathcal{G}_{2,4}$ are 8 point conditions for A_3.
Hence

$$N_{\mathcal{G}_{2,4}}(0,8) = 16695 = \text{number of braid arrangements}$$
$$\text{that contain 8 general points}$$

Cones of generic arrangements

An arrangement $\mathcal{A}$ of $k \geq n$ hyperplanes in $\mathbb{P}^n$ is called a generic d-cone if there is a linear space X of dimension d common to all the hyperplanes in $\mathcal{A}$ and if no point outside of X lies on more than n of the hyperplanes.

Any generic d-cone $\mathcal{A}$ is a cone over the generic arrangement in $\mathbb{P}^{n-d-1}$, obtained by replacing each hyperplane in $\mathbb{P}^{n-d-1}$ by the linear span of the hyperplane and X.

Generic d-cones in $\mathbb{P}^n$

Let $\mathcal{A}$ be a generic d-cone arrangement of k hyperplanes in $\mathbb{P}^n$.

Then $\mathcal{A}$ is determined by

1. $X \in \mathbb{G}(d, n)$=the Grassmanian of d-dimensional linear subspaces of $\mathbb{P}^n$
2. k points in $\mathbb{P}(\mathbb{C}^{n+1}/X) = \mathbb{P}^{n-d-1}$

$D = \dim \mathcal{M}(\mathcal{A}) = \mathbb{G}(d, n) \times (\mathbb{P}^{n-d})^k =$
$(d+1)(n-d) + k(n-d-1)$

In order to get $N_{\mathcal{A}}$ we will need to know how many ways there are to choose X and satisfy our point conditions.

This is exactly the subject of Schubert calculus.

Schubert Calculus

$H^*(\mathbb{G}(d,n),\mathbb{Z})$ is generated by σ_α where α is a $d+2$ tuple of non-increasing non-negative integers $\alpha_i \leq d-n$.

The products of these classes are given by the Pieri and Giambelli formulas.

If $|\alpha_1| + \cdots + |\alpha_t| = \dim \mathbb{G}(d,n) = (n-d)(d+1)$ then the product has well defined degree denoted $\int_{\mathbb{G}(d,n)} \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ which is the number of d planes in the intersection of the corresponding Schubert varieties.

- Let $(1,\ldots,1,0,\ldots,0) =: 1^i$ where there are i 1's.
- For $s = (s_0,\ldots,s_{d+1}) \in \mathbb{N}^{d+2}$ let

$$\sigma^s = \prod_{i=0}^{d+1} \sigma_{1^i}^{s_i}$$

Main theorem

Theorem (Paul, Traves, W.)

If $\mathcal{A}$ is a generic d-cone in $\mathbb{P}^n$ consisting of k hyperplanes then the the number of generic d-cones that pass through $D = (d+1)(n-d) + k(n-d-1)$ points in general position is $N_{\mathcal{A}} =$

$$\frac{\sum_{\Gamma} \sigma^{s} \binom{k}{s_0, s_1, \ldots, s_{d+1}} \binom{D}{(n)^{s_{d+1}}, (n-1)^{s_d}, \ldots, (n-(d+1))^{s_0}}}{k!},$$

where $\Gamma =$

$$\left\{ (s_0, \ldots, s_{d+1}) \in \mathbb{N}^{d+2} : \sum_{i=0}^{d+1} i s_i = \dim \mathbb{G}(d, n), \sum_{i=0}^{d+1} s_i = k \right\}$$

Mulţumesc!!!

generic 0-cones from $\mathbb{P}^1$ to $\mathbb{P}^2$

Theorem (Paul, Traves, W.)

If $\mathcal{A}$ is a generic 0-cone of k lines in $\mathbb{P}^2$ then $\dim M(\mathcal{A}) = k + 2$ *and the characteristic numbers are* $N_{\mathcal{A}}(k+2, 0) = 3\binom{k+2}{4}$, $N_{\mathcal{A}}(k+1, 1) = \binom{k+1}{2}$, $N_{\mathcal{A}}(k, 2) = 1$. *All other characteristic numbers are 0.*

Note: To be tangent to a line an intersection point of the arrangement must be on the line.

For a generic 0-cone to be tangent to 2 lines then the unique intersection point of the arrangement must be on the intersection point of the 2 lines. Then the *k*-points uniquely determine the arrangement.