BUNDLES OF HIGHER LOGARITHMIC DERIVATIONS

Jean Vallès
Université de Pau et des Pays de l'Adour
Meeting AMS-RMS 2013

Aim of this work (in progress):

(1) Associate, to any line arrangement in $\mathbb{P}^2$ and for any $d \geq 0$, a vector bundle of rank $d+1$ and a torsion sheaf supported by the $(d+2)$ multiple points of the arrangement.
For $d = 1$, recover the bundle of logarithmic derivations introduced by Saito in 1980 in *Theory of logarithmic differential forms and logarithmic vector*.
(2) Define freeness for these bundles (d-freeness) and give examples.
(3) Deduce (new) results for the bundle of logarithmic derivations ($d = 1$) and Terao's conjecture: "*Freeness depends only on combinatorics*".

Bundle of logarithmic derivations

$$\{L_1, \dots, L_m\} \rightsquigarrow D = \cup_{i=1}^{m} L_i = \{f = 0\}$$

Definition. The rank 2 vector bundle, kernel of the jacobian map

$$0 \longrightarrow T_D \longrightarrow \mathscr{O}_{\mathbb{P}^2}^3 \xrightarrow{(\partial_0,\partial_1,\partial_2)} \mathscr{O}_{\mathbb{P}^2}(m-1),$$

is called *bundle of logarithmic derivations*

Its sections $\delta \in \mathrm{H}^0(\mathbb{P}^2, T_D(n))$ are derivations

$$\delta = F_0\partial_0 + F_1\partial_1 + F_2\partial_2$$

where $F_i \in \mathbb{C}[X_0, X_1, X_2]_n$, such that $\delta(f) = 0$.

$$\left.\begin{array}{l} V = \mathrm{vect}\{X_0, X_1, X_2\} \\ V^* = \mathrm{vect}\{\partial_0, \partial_1, \partial_2\} \end{array}\right\} \Rightarrow \delta \in S^n V \otimes V^*.$$

Bundle of higher logarithmic derivations: A vector bundle parametrizing derivations ν such that

$$\nu \in S^n V \otimes S^d V^* \text{ and } \nu(f) = 0.$$

The good space to work on the incidence variety

$$\begin{array}{ccc} \mathbb{P}^2 & \xleftarrow{p} & \mathbb{I} = \{(x,y) | x \in L_y\} \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2 \\ & & \downarrow q \\ & & \check{\mathbb{P}}^2 \end{array}$$

$$\mathbb{I} = \{X_0\partial_0 + X_1\partial_1 + X_2\partial_2 = 0\}.$$

Here $\partial_0, \partial_1, \partial_2$ are dual coordinates!

$$p^{-1}(x) \simeq L_x \quad \text{and} \quad qp^{-1}(x) = L_x$$
$$q^{-1}(y) \simeq L_y \quad \text{and} \quad qp^{-1}(y) = L_y$$

$$x = (a, b, c) \leftrightsquigarrow L_x = \{a\partial_0 + b\partial_1 + c\partial_2 = 0\}$$

$$y = (\alpha, \beta, \gamma) \leftrightsquigarrow L_y = \{\alpha X_0 + \beta X_1 + \gamma X_2 = 0\}$$

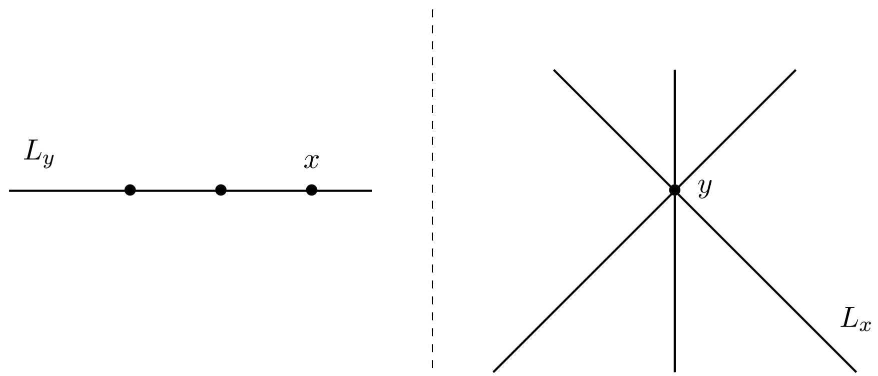


FIGURE 1. 3 aligned points - 3 concurrent lines

$$\begin{array}{ccc} D = \cup_{i=1}^{m} L_{y_i} \subset \mathbb{P}^2 & \leftrightsquigarrow & Z = \{y_1, \dots, y_m\} \subset \check{\mathbb{P}}^2 \\ \downarrow & & \downarrow \\ T_D & \overset{?}{\leftrightsquigarrow} & \mathcal{I}_Z \subset \mathcal{O}_{\check{\mathbb{P}}^2} \end{array}$$

Theorem (FMV, Compositio 2013)**.** $T_D = p_*(q^*\mathcal{I}_Z(1))$.

Proof. Recall that $\mathbb{I} = \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}(-1))$.
The canonical exact sequence twisted by 1:

$$0 \to \mathcal{I}_Z(1) \longrightarrow \mathcal{O}_{\check{\mathbb{P}}^2}(1) \longrightarrow \mathcal{O}_Z(1) \to 0,$$

gives on the $\mathbb{P}^2$ side (apply the functor p_*q^*):

$$0 \to p_*(q^*\mathcal{I}_Z(1)) \longrightarrow T_{\mathbb{P}^2}(-1) \xrightarrow{res} \oplus_{y_i \in Z} \mathcal{O}_{L_{y_i}}$$

“Unicity” of the map “res” $\Rightarrow p_*(q^*\mathcal{I}_Z(1)) = T_D$. ∎

Generalisation

The canonical exact sequence twisted by $d \geq 1$:

$$0 \to \mathcal{I}_Z(d) \longrightarrow \mathscr{O}_{\check{\mathbb{P}}^2}(d) \longrightarrow \mathscr{O}_Z(d) \to 0,$$

gives on the $\mathbb{P}^2$ side:

$$0 \to p_*(q^*\mathcal{I}_Z(d)) \longrightarrow \mathrm{Sym}^d(T_{\mathbb{P}^2}(-1)) \xrightarrow{res} \oplus_{y_i \in Z} \mathscr{O}_{L_{y_i}}$$

The kernel is unique (up to linear isomorphism)

Notations

- $T_Z^{(d)} := p_* q^* \mathcal{I}_Z(d)$ is a v. b. of rank $d+1$
- $\mathfrak{R}_Z^{(d)} := R^1 p_* q^* \mathcal{I}_Z(d)$ is a torsion sheaf supported by $\{x, |L_x \cap Z| \geq d+2\}$

These sheaves can be understood locally

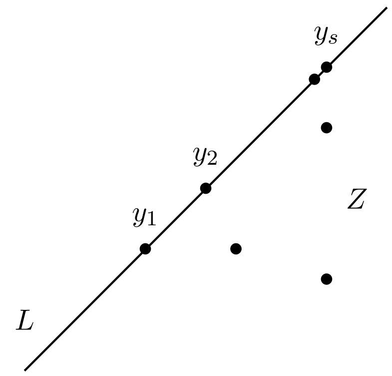


FIGURE 2. $|L \cap Z| = s$

$$\mathcal{I}_Z(d) \otimes \mathscr{O}_L = \mathscr{O}_L(d-s) \oplus \mathscr{O}_{y_1} \oplus \cdots \oplus \mathscr{O}_{y_s}$$

$$T_Z^{(d)} \otimes \mathbf{k}_x = \mathrm{H}^0(\mathcal{I}_Z(d) \otimes \mathscr{O}_{L_x}) \text{ constant}$$

$$\mathfrak{R}_Z^{(d)} \otimes \mathbf{k}_x = \mathrm{H}^1(\mathcal{I}_Z(d) \otimes \mathscr{O}_{L_x}) = \mathrm{H}^0(\mathscr{O}_L(s-d-2))$$

Let us summarize

$$D = \cup_{i=1}^{m} L_{y_i} \subset \mathbb{P}^2 \quad \leftrightsquigarrow \quad Z = \{y_1, \dots, y_m\} \subset \check{\mathbb{P}}^2$$

$$\mathscr{O}_{\mathbb{P}^2}(-D),\ \mathrm{sing}(D) \quad \leftrightsquigarrow \quad \mathcal{I}_Z$$

$$T_D, \text{triple points of } D \quad \leftarrow\!-\,- \quad \mathcal{I}_Z(1)$$

$$\{T_Z^{(d)}, \mathfrak{R}_Z^{(d)}\} \quad \leftarrow\!-\,- \quad \mathcal{I}_Z(d), d \geq 0$$

Example. Torelli problem: $\leftarrow\!-\,-$ or $\leftrightsquigarrow$?

Why it is interesting? Because any canonical exact sequence related to $\mathcal{I}_Z(d)$ gives an information on $\{T_Z^{(d)}, \mathfrak{R}_Z^{(d)}\}$.

Example.

$$(1) \quad 0 \to \mathcal{I}_Z(d) \longrightarrow \mathcal{I}_{Z\setminus\{y\}}(d) \longrightarrow \mathcal{O}_y \to 0$$

$$(2) \quad 0 \to \mathcal{I}_{Z_1}(d-1) \longrightarrow \mathcal{I}_Z(d) \longrightarrow \mathcal{O}_{L_x}(d - |Z \cap L_x|) \to 0$$

$$(3) \quad 0 \to \mathcal{I}_Z(d-1) \longrightarrow \mathcal{I}_Z(d) \longrightarrow \mathcal{I}_Z(d) \otimes \mathcal{O}_L \to 0$$

(1) Ziegler Addition-Deletion Theorem, and the number of triple points on L_y.
(2) Combinatorics of subarrangements.
(3) An injective map $T_Z^{(d-1)} \hookrightarrow T_Z^{(d)}$.

One can associate to Z many tuples of positive integers:

$$Z \rightsquigarrow (a_{1,1}, a_{1,2}); (a_{2,1}, a_{2,2}, a_{2,3}); (a_{3,1}, a_{3,2}, a_{3,3}, a_{3,4}); \ldots$$

such that

$$a_{1,1} \leq a_{1,2} \text{ and } a_{1,1} + a_{1,2} = m - 1,$$

$$a_{2,1} \leq a_{2,2} \leq a_{2,3} \text{ and } a_{2,1} + a_{2,2} + a_{2,3} = m - 3,$$

$$a_{d,1} \leq a_{d,2} \leq \cdots \leq a_{d,d+1} \text{ and } a_{d,1} + a_{d,2} + \cdots + a_{d,d+1} = m - \binom{d+1}{2}.$$

Indeed $T_Z^{(d)} \otimes \mathscr{O}_L = \mathscr{O}_L(-a_{d,1}) \oplus \cdots \oplus \mathscr{O}_L(-a_{d,d+1})$ for L general.

Geometric invariants of Z, ie $\mathrm{H}^0(\mathcal{I}_Z \otimes \mathcal{I}_x^{a_{d,1}}(d + a_{d,1})) \neq 0$.

Question Do we always have $a_{d,i} \leq a_{d+1,j}$?

Example

Dual Hesse 9 lines dual to the 9 inflection points of a smooth cubic curve C.

Proposition.

$$T_Z = \mathscr{O}^2_{\mathbb{P}^2}(-4), \quad T_Z^{(2)} = \mathrm{Sym}^2(\Omega_{\mathbb{P}^2})(1)$$

Proof.

$$Z = C \cap \mathrm{Hessian}(C) \;\Rightarrow\; \mathscr{O}_C(-Z) = \mathscr{O}_C(-3).$$

$$0 \to \mathscr{O}_{\check{\mathbb{P}}^2}(-1) \longrightarrow \mathcal{I}_Z(2) \longrightarrow \mathscr{O}_C(-1) \to 0.$$

$$p_* q^* \mathscr{O}_{\check{\mathbb{P}}^2}(-1) = R^1 p_* q^* \mathscr{O}_{\check{\mathbb{P}}^2}(-1) = 0$$

$$\Downarrow$$

$$T_Z^{(2)} = p_* q^* \mathscr{O}_C(-1) = \mathrm{Sym}^2(\Omega_{\mathbb{P}^2})(1).$$

(In particular $T_Z^{(2)} \not\rightsquigarrow Z$, i.e. not Torelli) $\square$

On any line l we have

$$\mathrm{Sym}^2(\Omega_{\mathbb{P}^2})(1) \otimes \mathcal{O}_l = \mathcal{O}_l(-1) \oplus \mathcal{O}_l(-2) \oplus \mathcal{O}_l(-3).$$

It implies that there is, at any point, a cubic curve passing through Z and this point.

In other words there is a pencil of cubics through Z.

If Z consists of 9 points not on a pencil, then on the general line l:

$$T_Z^{(2)} \otimes \mathcal{O}_l = \mathcal{O}_l(-2) \oplus \mathcal{O}_l(-2) \oplus \mathcal{O}_l(-2).$$

Remark. If $Z \subset C_{d+1}$ then

$$T_Z^{(d)} = p_* q^* \mathcal{O}_{C_{d+1}}(-Z)$$

In particular Z is not d-Torelli.

Resolution of BHD

$$0 \to \mathscr{O}_{\mathbb{P}^2\times\check{\mathbb{P}}^2}(-1,-1) \xrightarrow{\sum X_i Y_i} \mathscr{O}_{\mathbb{P}^2\times\check{\mathbb{P}}^2} \longrightarrow \mathscr{O}_{\mathbb{I}} \to 0.$$

Tensor by $q^*\mathcal{I}_Z(d)$ and push down by p:

$$0 \to \mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(d-1))\otimes\mathscr{O}_{\mathbb{P}^2}(-1) \to \mathrm{H}^0(\check{\mathbb{P}}^2, \mathcal{I}_Z(d))\otimes\mathscr{O}_{\mathbb{P}^2} \to T_Z^{(d)} \to\to \mathrm{H}^1(\check{\mathbb{P}}^2, \mathcal{I}_Z(d-1))\otimes\mathscr{O}_{\mathbb{P}^2}(-1) \to \mathrm{H}^1(\check{\mathbb{P}}^2, \mathcal{I}_Z(d))\otimes\mathscr{O}_{\mathbb{P}^2} \to \mathfrak{R}_Z^{(d)} \to 0.$$

When $Z \not\subset$ (curve of degree d), we have

$$0 \to T_Z^{(d)} \to (\mathscr{O}_{\mathbb{P}^2}(-1))^{m-\binom{d+1}{2}} \to \mathscr{O}_{\mathbb{P}^2}^{m-\binom{d+2}{2}} \to \mathfrak{R}_Z^{(d)} \to 0.$$

When there is no $d+2$ multiple point,

$$0 \to T_Z^{(d)} \to \mathscr{O}_{\mathbb{P}^2}^{m-\binom{d+1}{2}}(-1) \to \mathscr{O}_{\mathbb{P}^2}^{m-\binom{d+2}{2}} \to 0.$$

Chern classes: $c_1(T_Z^{(d)}) = \binom{d+1}{2} - m$, $c_2(T_Z^{(d)}) = \binom{m-\binom{d+1}{2}}{2} - |\mathfrak{R}_Z^{(d)}|$.

Freeness and d-freeness

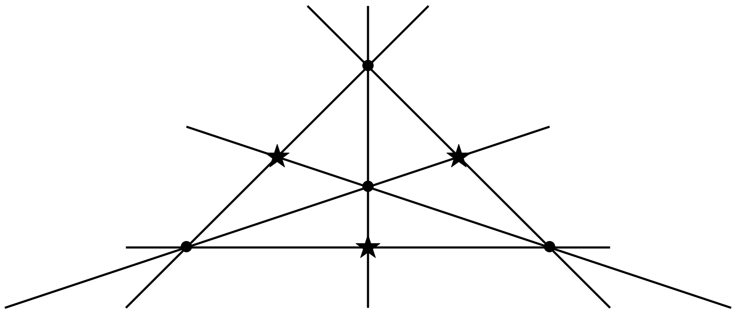
Definition. D is d-free (free=1-free) with exponents $(a_1, \dots, a_{d+1}) \in \mathbb{N}^{d+1}$ when

$$T_Z^{(d)} = \mathscr{O}_{\mathbb{P}^2}(-a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^2}(-a_{d+1}).$$

- If D is d-free, $|\mathfrak{R}_Z^{(d)}| = \binom{m - \binom{d+1}{2}}{2} - \sum_{i<j} a_i a_j$.

A general finite Z is not free, on the contrary it leads to stable vector bundles.

Freeness and d-freeness: Example



Ceva's arrangement:
six lines, 4 triple points $\bullet$, 3 double $\bigstar$

- $T_Z = \mathscr{O}_{\mathbb{P}^2}(-2) \oplus \mathscr{O}_{\mathbb{P}^3}(-3)$;
- $T_Z^{(2)} = \mathscr{O}_{\mathbb{P}^2}^3(-1)$ ($Z \not\subset$ conic and $m = 6$).

Freeness and d-freeness: Example

Hesse arrangement 12 lines through 9 inflection points.
9 quadruple points $\bullet$, 12 double points $\bigstar$.

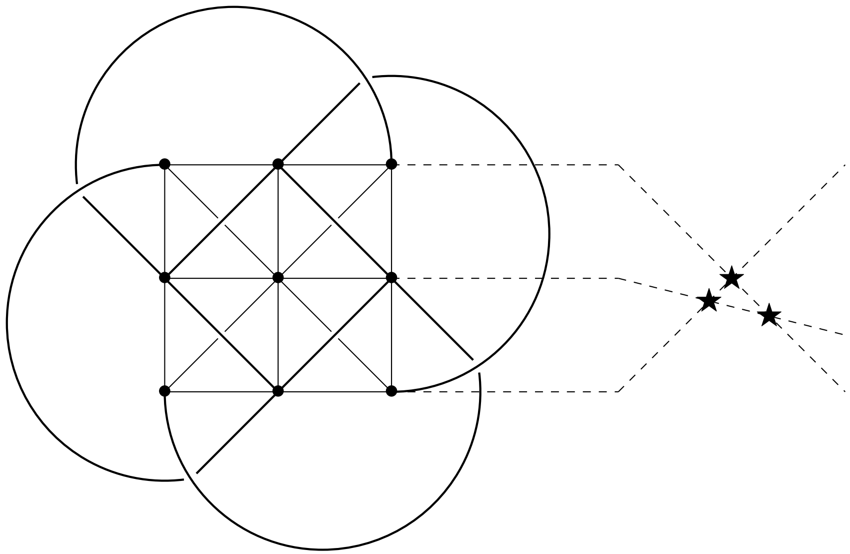


FIGURE 3. Hesse arrangement

$$T_Z = \mathscr{O}_{\mathbb{P}^2}(-4) \oplus \mathscr{O}_{\mathbb{P}^2}(-7), \quad T_Z^{(2)} \stackrel{?}{=} \mathscr{O}^3_{\mathbb{P}^2}(-3), \quad T_Z^{(3)} \stackrel{?}{=} \Omega \oplus \Omega$$

Freeness and d-freeness: Example

B3 arrangement: 9 lines, 3 quadruple points and 4 triple points

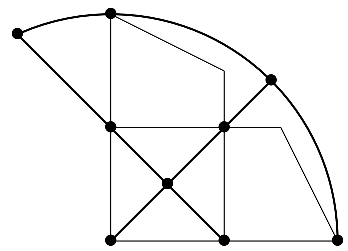


FIGURE 4. Dual picture of B3 arrangement: 9 points, 7 lines

- $T_Z = \mathscr{O}_{\mathbb{P}^2}(-3) \oplus \mathscr{O}_{\mathbb{P}^3}(-5)$;
- $T_Z^{(2)} = \mathscr{O}_{\mathbb{P}^2}^3(-2)$;
- $T_Z^{(3)} = \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}^3(-1)$, since $\mathrm{H}^0(\mathcal{I}_Z(3)) \neq 0$.

An arrangement closed to B3: 9 lines, 3 quadruple points and no triple point.

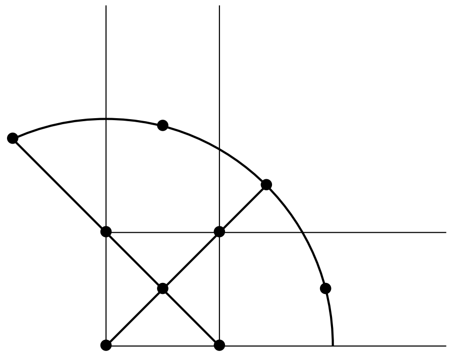


FIGURE 5. Dual picture: 9 points on 3 four secant lines

- T_Z is not free;
- $T_Z^{(2)} = \mathscr{O}_{\mathbb{P}^2}^3(-2)$;
- $T_Z^{(3)} = \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}^3(-1)$, since $\mathrm{H}^0(\mathcal{I}_Z(3)) \neq 0$.

Z on d lines: d-freeness

B3, quite B3 and Ceva arragements are consequences of:

Theorem. *Assume that* $Z \subset \cup_{i=1}^{d+1} L_i$, $\cup L_i$ *has no triple point,* $L_i \cap L_j \in Z$ *and* $|L_i \cap Z| = n_i + d, n_i \geq 0$. *Then* $T_Z^{(d)} = \oplus_{i=1}^{d+1} \mathscr{O}_{\mathbb{P}^2}(-n_i)$.

Proof. The map $\mathcal{I}_Z(d) \to \oplus \mathscr{O}_{L_i}(-n_i)$ is surjective. Its kernel is $\mathscr{O}_{\check{\mathbb{P}}^2}(-1)$. It has nor direct image neither higher direct image. ∎

Find d-free arrangements by induction

Theorem. *Let Z be a set of m points, $t \geq 0$ and L be a line such that $|L \cap Z| = d + t$. Assume that any k secant line ($k \geq d+1$) to $Z_1 = Z \setminus L$ is $k+1$ secant to Z. Then, Z_1 is $(d-1)$-free w.e. $(a_1, \ldots, a_d)$ implies Z is d-free w.e. $(a_1, \ldots, a_d, t)$.*

Let Z be d-free. Let L a line meeting each $d+2$ secant l_i to Z in a point y_i, $1 \leq i \leq s$ then $Z \cup \{y_1, \ldots, y_s\}$ is d-free.

Find d-free arrangements by induction. Hint of the proof

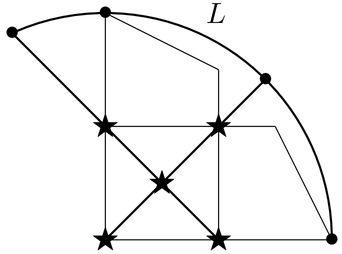


FIGURE 6. ★: Points of Z_1

$$0 \to \mathcal{I}_{Z_1}(1) \to \mathcal{I}_Z(2) \to \mathscr{O}_L(-2) \to 0$$

gives on the other side

$$0 \to T_{Z_1} \to T_Z^{(2)} \to \mathscr{O}_{\mathbb{P}^2}(-2) \to \mathfrak{R}_{Z_1}^{(1)} \to \mathfrak{R}_Z^{(2)} \to \mathscr{O}_{L^\vee} \to 0.$$

By the previous theorem $T_{Z_1} = \mathscr{O}_{\mathbb{P}^2}^2(-2)$. By hypothesis we have two short e.s. Then,

$$T_Z^{(2)} = \mathscr{O}_{\mathbb{P}^2}^3(-2).$$

Terao's conjecture for line arrangements

Conjecture. Assume that D_0 (or Z_0) is free with exponents $(n, n+r)$ and that D (or Z) has the same combinatorics. Then Z is free w.e. $(n, n+r)$.

Theorem. *Terao's conjecture is true for arrangements of m lines such that $c_1(T_Z^{(2)}) = 3 - m$ and $c_2(T_Z^{(2)}) = \binom{m-3}{2}$.*

Proof. These Chern classes implies that D does no contain any quadruple points. Then, it's a consequence of a theorem proved by W-Y[1]. □

---

[1] Wakefield, Yuzvinsky: Derivations of an effective divisor on the complex projective line. Trans. Amer. Math. Soc., 2007

Terao's conjecture for line arrangements. Link between $d+2$ multiple points and exponents

Denote by $t_{d+2,L}$ the number (counted with multiplicity) of $d+2$ multiple points of D living on L, where L is a line of the arrangement.

Theorem. *Assume that D has the combinatorics of a d-free arrangement w.e. $(a_1,\dots,a_{d+1})$. Then, for any line of the arrangement,*

$$t_{d+2,L} \geq \min\{a_i|1 \leq i \leq d+1\} - 1.$$

Proof. Let $x \in Z$ corresponding to the line L_x of the arrangement. Send the following exact sequence

$$0 \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z\setminus\{x\}}(d) \to \mathscr{O}_x \to 0$$

on the other side. $\square$

Questions

(1) Characterize arrangements that are d-free for any d.
(2) Terao's conjecture for d-free arrangements, any $d \geq 1$.
(3) Terao's conjecture for d-free arrangements, one $d \geq 1$.
(4) Does the splitting of $T_Z^{(d)}$ depends on curves of degree d through subsets of points of Z?
(5) Torelli problem for the sheaves $T_Z^{(d)}$.