Elliptic surfaces
and
Zariski pairs for conic-line arrangements

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Motivation and Background

- S: a set of finite number primes $\{p_1, \ldots, p_s\}$.

  Study Galois extensions $K/\mathbb{Q}$ such that $\mathcal{O}_K$ is ramified at most over S, where $\mathcal{O}_K$ is the ring of integers in K. And how primes $q (\notin S)$ behaves $(\langle q\rangle_{\mathcal{O}_K} = \mathfrak{q}_1 \cdots \mathfrak{q}_t)$

- $D_1, \ldots, D_s$: irreducible curves on $\mathbb{P}^2$.

  Study Galois extensions $K/\mathbb{C}(x, y)$ such that the normalization of $\mathbb{P}^2$ in K gives rise to Galois covers of $\mathbb{P}^2$ $(\pi : X \longrightarrow \mathbb{P}^2)$ ramified over at most $D_1 \cup \ldots \cup D_s$. And how other curves $C (\neq D_i)$ behaves $(\pi^*C = C_1 + \cdots + C_r$, properties of $C_1, \ldots, C_r)$

'number theory' over $\mathbb{C}(x, y)$

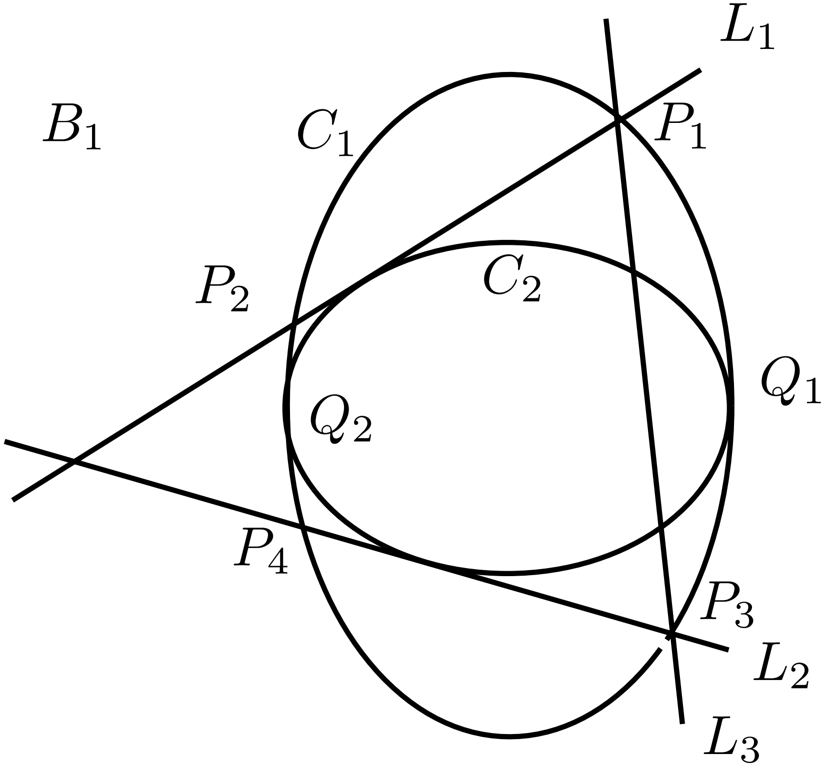
In this talk

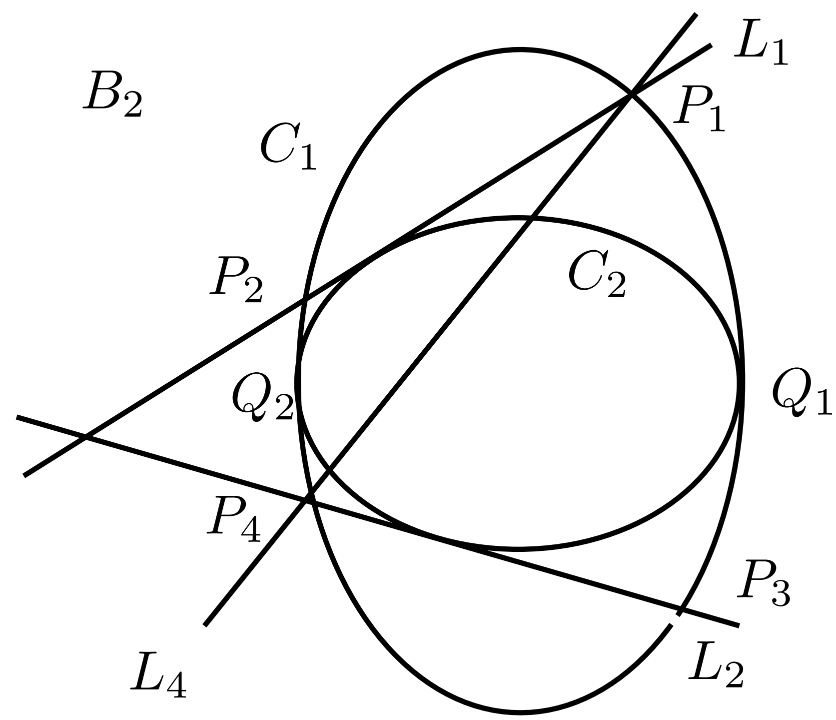
- Geometry and arithmetic of sections of elliptic surfaces
- Double covers of $\mathbb{P}^2$
- Study on Galois covers of $\mathbb{P}^2$ with given branch set. In our case, the Galois group is isomorphic to the dihedral group D_{2p} of oder $2p$, p: odd prime
- Applications: **Zariski pair for conic-line arrangement** and Zarisk N-tuple for conic arrangements (with S. Bannai)

We explain our approach through an example:

Example

Consider two conic-line arrangements B_1 and B_2 in $\mathbb{P}^2$ as follows:





Theorem

Let B_1 and B_2 be the conic-line arrangements as in the previous slide. Then

$$\nexists \text{ homeomorphism } h : \mathbb{P}^2 \to \mathbb{P}^2 \text{ such that } h(B_1) = B_2.$$

i.e., (B_1, B_2) *is a Zariski pair*

Elliptic surfaces

Elliptic surface S: a smooth projective fibered surface $\varphi : S \to C$ over a smooth projective curve, C, as follows:

(i) $\exists$ finite subset, $\mathrm{Sing}(\varphi) \neq \emptyset \subset C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 (resp. a singular curve) for $v \notin \mathrm{Sing}(\varphi)$ (resp. $v \in \mathrm{Sing}(\varphi)$).

(ii) $\exists\ O : C \to S$ (we identify O with its image).

(iii) $\nexists$ exceptional curve of the first kind in any fiber.

In our case: $C = \mathbb{P}^1$.

Mordell-Weil group 1

MW(S): the set of sections of S. (We identify a section with its image on S.)

1. MW(S) can be regarded as an Abelian group under fiberwise addition, O being the zero element.

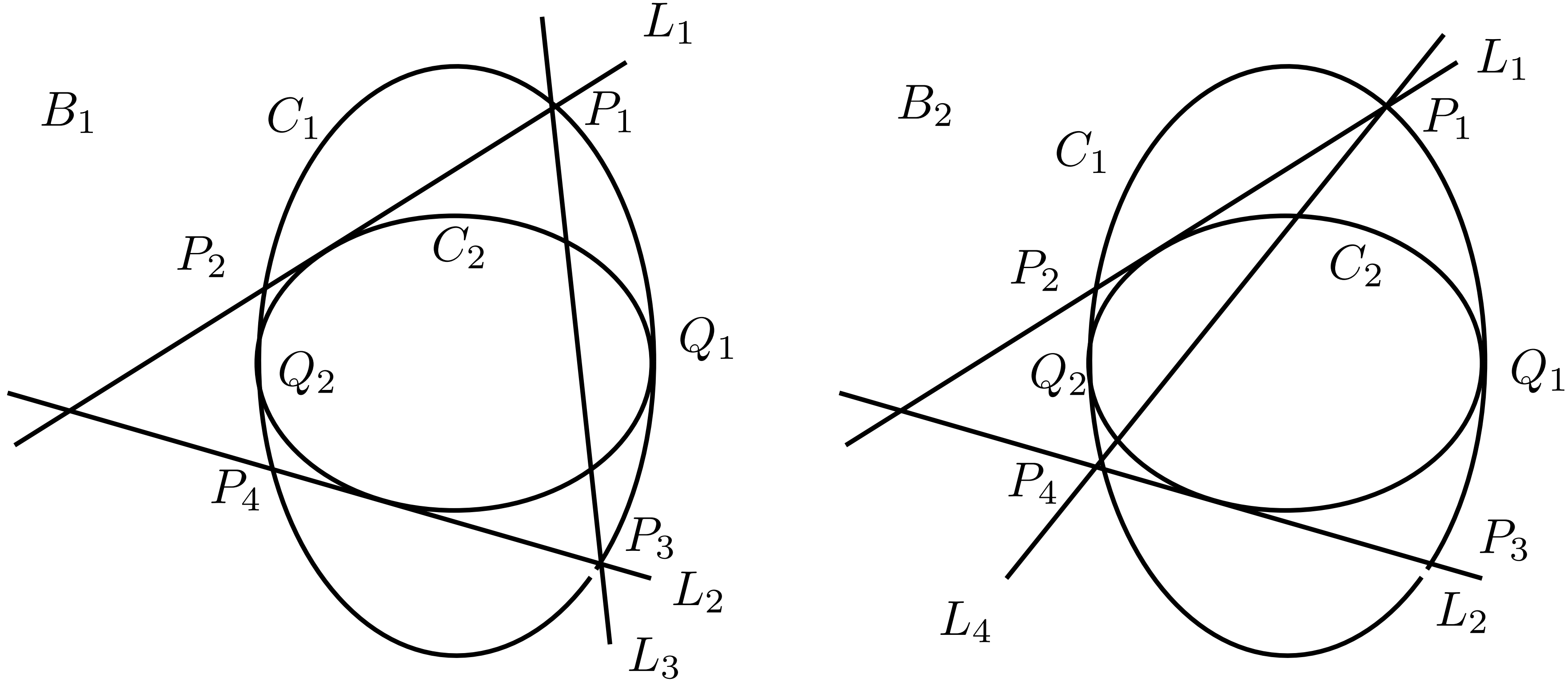
2. MW(S) is called the Mordell-Weil group of $\varphi : S \to \mathbb{P}^1$. Under our assumption, MW(S) is finitely generated (T. Shioda).

Mordell-Weil group 2

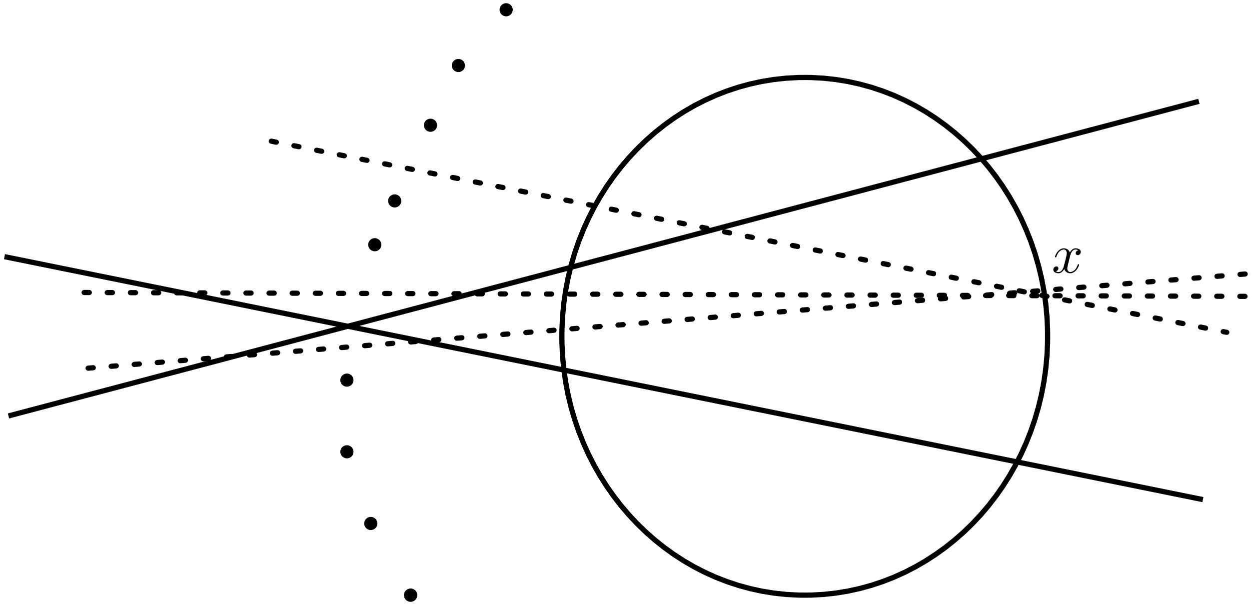
$\dot{+}$: group law on $\mathrm{MW}(S)$.

$[m]s$: the multiplication-by-m map ($m \in \mathbb{Z}$) on $\mathrm{MW}(S)$ for $s \in \mathrm{MW}(S)$.

Given $s_1, \dots, s_k \in \mathrm{MW}(S) \Rightarrow \sum_i [a_i] s_i$ another element of $\mathrm{MW}(S)$, a new curve on S.


B_1
L_1
C_1
P_1
P_2
C_2
Q_1
Q_2
P_4
P_3
L_2
L_3
B_2
L_1
P_1
C_1
P_2
C_2
Q_2
Q_1
P_4
P_3
L_4
L_2

- $\mathcal{Q} := C_1 + L_1 + L_2$. e.g. $\mathcal{Q}: (x-t^2)(x-3t+2)(x+3t+2)$ $= f(x,t)$
- $f'' : S'' \to \mathbb{P}^2$: double cover with the branch locus $\Delta_{f''} = \mathcal{Q}$.
  e.g. $y^2 = f(x,t)$
- x: a general point of C_1; and the pencil of lines through x.
- Λ_x: the pencil of lines through x; and Λ_x gives rise to a pencil of curves of genus 1, $\widetilde{\Lambda}_x$, on S''.
- Resolve singularities of S'' and the base points of $\widetilde{\Lambda}_x$; and we denote the obtained surface by S and the resolution map by $\overline{\mu}$.
- $\varphi : S \to \mathbb{P}^1$ is induced by the pencil $\widetilde{\Lambda}_x$.


x

How to obtain B_1 and B_2

How do we obtain C_2?

- L_3 and L_4 give rise to sections $s^{\pm}_{L_3}$ and $s^{\pm}_{L_4}$ on S.
- $\overline{\mu} \circ f''([2]s^{\pm}_{L_3})$ and $\overline{\mu} \circ f''([2]s^{\pm}_{L_4})$ are both smooth conics as in C_2 (i.e., inscribing $C_1 + L_1 + L_2$).
- We may assume that $C_2 = \overline{\mu} \circ f''([2]s^{\pm}_{L_3})$.

One can see 'difference' between B_1 and B_2 in S, not in $\mathbb{P}^2$!

Key Theorem

$s_1, s_2 \in \mathrm{MW}(S)$.

There exists a Galois cover of $\mathbb{P}^2$ such that

- the Galois group is isomorphic to D_{2p},
- the ramification indices along:

  C_1, L_1 and $L_2 = 2$,

  $\overline{\mu} \circ f''(s_i) = p$ $(i = 1, 2)$

$\Leftrightarrow$ s_1 and s_2 give linearly dependent elements in $\mathrm{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z}$.

Remark. Key Theorem holds under more general setting.

Theorem follows from Key Theorem immediately as follows:

- $\{s^+_{L_3}, s^+_{L_4}\}$ forms a basis of the free part of $\mathrm{MW}(S)$.
- B_1: Put $s_1 = s^+_{L_3}, s_2 = [2]s^+_{L_3}$.

  There exists a Galois cover of $\mathbb{P}^2$ such that

  (i) the Galois group is isomorphic to D_{2p},

  (ii) the ramification indices along

  C_1, L_1 and $L_2 = 2$,

  $L_3, C_2 = p$.

  B_2: Put $s_1 = s^+_{L_4}, s_2 = [2]s^+_{L_3}$.

  There exists **no** Galois cover of $\mathbb{P}^2$ such that

  (i) the Galois group is isomorphic to D_{2p},

  (ii) the ramification indices: along C_1, L_1 and $L_2 = 2$; and along $L_4, C_2 = p$.

Rem: $\nexists\ \pi_1(\mathbb{P}^2 \setminus B_{2,+}) \longrightarrow D_{2p}$

Thank you!