Graphs, groups, homology

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Toric complexes

Torus: real $T^n = (\mathbb{S}^1)^n$, complex $\mathbb{T}^n = (\mathbb{C}^*)^n$.

Δ_Γ flag complex associated to $\Gamma = (V = [n], E)$ simplicial graph:

r-simplex σ of $\Delta_\Gamma \longleftrightarrow$ complete subgraph K_{r+1} of Γ

Definition

$T_\Gamma \subset T^n$ or $\mathbb{T}_\Gamma \subset \mathbb{T}^n$ toric complex associated to Γ:

$$\bigcup_{\sigma \in \Delta} \{z_v = 1 \text{ if } v \notin \sigma\}.$$

T_Γ minimal cell structure: r-cells $c_\sigma \longleftrightarrow (r-1)$-simplices σ of Δ_Γ.

$T_\Gamma \hookrightarrow \mathbb{T}_\Gamma$ strong deformation retract.

Right-angled Artin groups

Definition

G_Γ right-angled Artin group associated to Γ:

$$G_\Gamma = \langle g_v, v \in V \mid [g_u, g_v] = 1, uv \in E \rangle.$$

[Charney-Davis] $G_\Gamma = \pi_1(T_\Gamma)$, and T_Γ is aspherical.

[Kim-Rousch] $H^*(T_\Gamma, \mathbb{K}) = \mathbb{K}\langle\Delta\rangle$ Stanley-Reisner ring.

Example

- $\Gamma = K_n$ complete graph, $\Delta_\Gamma = \Delta^{n-1}$, $T_\Gamma = T^n$.
- $\Gamma = \overline{K}_n$ no edges graph, $\Delta_\Gamma = \Gamma$, $T_\Gamma = \bigvee^n S^1$.

Artin kernels

$\chi : G_\Gamma \to \mathbb{Z}$ epimorphism, $\chi(g_v) = d_v \neq 0$.

Definition ([Papadima-Suciu])

$N_\Gamma^\chi := \ker(\chi)$ right-angled Artin kernel associated to Γ and χ.

[Bestvina-Brady]: $N_\Gamma := \ker(\chi)$, $\chi(g_v) = 1$ for $v \in V$.

[B-B][B-G][M-M-vW][P-S]:

- N_Γ^χ is finitely presented if and only if Δ_Γ is simply-connected.
- N_Γ^χ is of type F_r (resp. FP_r) if and only if Δ_Γ is $(r-1)$-connected.
- N_Γ^χ is of type FP_r if and only if Δ_Γ is $(r-1)$-acyclic, if and only if $\dim_{\mathbb{K}} H_{\leq r}(N_\Gamma^\chi, \mathbb{K}) < \infty$, for any field $\mathbb{K}$.

[Artal-Cogolludo-M.]: If Γ connected tree on n vertices then N_Γ^χ is a free group of rank $1 + \sum_{v\in V}(n_v - 1)d_v$.

$\mathbb{Z}$-covers

$\mathbb{K}$ field, $\Lambda = \mathbb{K}[t^{\pm 1}]$ Laurent ring.

- $C_*(\widetilde{T_\Gamma}, \mathbb{K})$ equivariant chain complex of universal cover $\widetilde{T_\Gamma}$.
- Koszul differential on $C_r(\widetilde{T_\Gamma}, \mathbb{K}) = \mathbb{K}G_\Gamma \otimes_{\mathbb{K}} C_r(T_\Gamma, \mathbb{K})$:

$$\widetilde{\partial}_r(1 \otimes c_\sigma) = \sum_{i=1}^{r} (-1)^{i-1} (g_{v_i} - 1) \otimes c_{\sigma \setminus \{v_i\}},$$

  for $\sigma = \langle v_1, \ldots, v_r \rangle$ an $(r-1)$-simplex.
- $T_\Gamma^\chi \to T_\Gamma$ infinite cyclic cover associated to $\ker(\chi)$; $T_\Gamma^\chi = K(N_\Gamma^\chi, 1)$.
- $C_*(T_\Gamma^\chi, \mathbb{K})$ chain complex of free Λ-modules, induced differential on basis $C_r(T_\Gamma^\chi, \mathbb{K})$ in bijection with $(r-1)$-simplices of Δ_Γ.
- $H_*(N_\Gamma^\chi, \mathbb{K}) = H_*(T_\Gamma^\chi, \mathbb{K}) = H_*(T_\Gamma, \mathbb{K}\mathbb{Z}_\chi)$ as Λ-modules.

Problem

Structure of $H_*(N_\Gamma^\chi, \mathbb{K})$ as $\mathbb{K}\mathbb{Z}$-module (Λ-module).

Monodromy

Γ connected graph, $\mathbb{K}$ field of characteristic zero.

[Papadima-Suciu]

1. Triviality tests for $H_{\leq r}(N_\Gamma^\chi, \mathbb{K})$.
2. N_Γ Bestvina-Brady case: $H_{\leq r}(N_\Gamma, \mathbb{K})$ trivial $\mathbb{K}\mathbb{Z}$-module iff Δ_Γ is $(r-1)$-acyclic.

$$H_i(N_\Gamma, \mathbb{K}) = \Lambda^{\dim \widetilde{H}_{i-1}(\Delta,\mathbb{K})} \oplus (\Lambda/(t-1))^{\dim B_{i-1}(\Delta,\mathbb{K})}$$

Problem

Non-triviality and non-semisimplicity of $H_i(N_\Gamma^\chi, \mathbb{K})$ as Λ-module.

Example

Γ tree on 4 vertices: $v_1 — v_2 — v_3 — v_4$, and $\chi = (d_1, d_2, d_3, d_4)$;
characteristic polynomial of monodromy $(t-1)(t^{d_2}-1)(t^{d_3}-1)$.
If $d | \gcd(d_1, d_2, d_3)$ but $d \not| d_4$ then $H_1(N_\Gamma^\chi, \mathbb{K})$ semi-simple.
If $d | \gcd(d_2, d_3)$ but $d \not| d_1, d_4$ then $H_1(N_\Gamma^\chi, \mathbb{K})$ NOT semi-simple.

Monodromy in H_1

Theorem ([Artal-Cogolludo-M.])

Γ connected simplicial graph, $\chi : G_\Gamma \to \mathbb{Z}$ epimorphism, $\chi(g_v) = d_v \neq 0$.

$$H_1(N_\Gamma^\chi, \mathbb{K}) = \prod_{d\in\mathbb{N}} \prod_{j\in\mathbb{N}} \left(\frac{\Lambda}{\Phi_d(t)^j} \right)^{e_j(d)},$$

where $\Phi_d(t)$ cyclotomic polynomial of order d, and

1. $e_j(d) = 0$ *if* $j > 2$,
2. $e_1(d) + 2e_2(d)$ *is equal to* $\widetilde{b}_0(\Gamma_d^{(0)}) + \widetilde{b}_0(\Gamma_d^{(1)}) - \mu_\Gamma + \delta_{1,d}$,
3. $e_2(d)$ *is equal to* $\operatorname{rank}\left(\widetilde{H}_0(V_d^{(0)}, \mathbb{K}) \to \widetilde{H}_0(\Gamma_d^{(1)}, \mathbb{K})\right)$.

χ-Weighted subgraphs

χ-weights on Γ:

- vertices $v \in V$ $\mu_v = 1$ if $d | d_v$, $\mu_v = 0$ otherwise.
- edges $e = uv \in E$ then $\mu_e = \mu_u + \mu_v$.

Subgraphs of Γ:

- $\Gamma_d^{(0)}$ on vertices V and edges $\{e \in E \mid \mu_e = 0\}$.
- $V_d^{(0)} = \{v \in V \mid \mu_v = 0\}$.
- $\Gamma_d^{(1)}$ on vertices V and edges $\{e \in E \mid \mu_e \leq 1\}$.

Proof.
Analysis of boundary map ∂_2^χ in $C_*(T_\Gamma^\chi, \mathbb{K}) = C_*(T_\Gamma, \mathbb{K}\mathbb{Z}_\chi)$. Explicit computation of Fitting ideals of Λ-module $\operatorname{coker}(\partial_2^\chi)$, inspired by unpublished result of [Papadima-M.]. □