On the second nilpotent quotient of higher homotopy groups, for hypersolvable arrangements

Daniela Anca Macinic,
"Simion Stoilow" Institute of the Romanian Academy,
University of Bucharest

Let $\mathcal{A}$ be a central, essential arrangement of hyperplanes in an affine complex vector space and let X be its complement in the ambient space.

Denote by $L(\mathcal{A})$ the *intersection lattice*, i.e. the partial ordered set of all intersections of various hyperplanes, ordered by reversed inclusion.

Question: When is a topological invariant of the complement X *combinatorial*, i.e. determined by the combinatorics of the lattice?

The rank of an arrangement, i.e. the codimension of the center of $\mathcal{A}$ is combinatorial.

(Rybnikov) $\pi_1(X)$ is not combinatorial.

– *generic arrangements*: arrangements in $\mathbb{C}^l$ such that the hyperplanes in any subarrangement of cardinality l are independent; $\pi_1(X)$ is free abelian in $|\mathcal{A}|$ generators (Hattori).

–*split-solvable arrangements* (Choudary-Dimca-Papadima): arrangements in $P^2\mathbb{C}$ having a very simple combinatorics; only double points except on the line at infinity; the fundamental group is a product of free groups.

–*fiber-type (supersolvable) arrangements* (Falk-Randell): their complements are at the top of a tower of fibrations, with fibers $\mathbb{C} - \{\textit{finite number of points}\}$; the fundamental group of the complement is an iterated almost direct product of free groups *hypersolvable arrangements* (Jambu-Papadima): the definition is combinatorial; some conditions on the lattice up to rank 2 are given;

- the hypersolvable class contains all previously described types of arrangements;
- they are generic sections of fiber-type arrangements; any hypersolvable arrangement can be "deformed" to a supersolvable one, without changing the collinearity relations, hence without changing the fundamental group;
- the fundamental group is again an almost direct product of free groups.

Question: How about the higher homotopy groups of the complement?

- not all arrangement complements are $K(\pi, 1)$ spaces;
- inside the hypersolvable class, the property of being $K(\pi, 1)$ is combinatorial; a hypersolvable arrangement is $K(\pi, 1)$ iff it is supersolvable;
- extending the results of Hattori on generic arrangements to the hipersolvable class, Papadima-Suciu showed that the higher homotopy groups of the complement vanish up to some combinatorially determined range p;
- $\pi_1(X)$ acts on the higher homotopy groups; hence $\pi_p(X)$ has a structure of $\mathbb{Z}\pi_1$-module;
- an explicit presentation of the $\pi_p(X)$ as $\mathbb{Z}\pi_1$-module is given;

Definition A space is called minimal if it has the homotopy type of a connected finite type CW-complex such that the betti number b_k is equal to the number of k-cells.

Example Complements of hyperplane arrangements are minimal spaces.

Let $\hat{\mathcal{A}}$ be the supersolvable deformation of the hypersolvable arrangement $\mathcal{A}$; then the complement Y of $\hat{\mathcal{A}}$ is $K(\pi_1(X), 1)$.

Define

$$p(X) := sup\{q| \ b_r(X; \mathbb{Q}) = b_r(K(\pi_1(X), 1); \mathbb{Q}), \forall r \leq q\}$$

the *order of π_1-connectivity* of a space X, having the homotopy type of a connected finite type CW complex.

$$2 \leq p \leq rank(\mathcal{A}) - 1$$

Theorem (Papadima-Suciu) Let $\mathcal{A}$ be a hypersolvable arrangement with complement X, fundamental group π and order of connectivity p. Then:
(1) X aspherical iff $\mathcal{A}$ supersolvable iff $p = \infty$
(2) If $p < \infty$, then the first non-vanishing higher homotopy group of X is $\pi_p(X)$

$\mathrm{gr}_\bullet(\pi_1) \otimes \mathbb{Q}$ is combinatorial

Let I be the augmentation ideal of the group ring $\mathbb{Z}\pi_1(X)$ and $\mathrm{gr}_I^\bullet \mathbb{Z}\pi_1$ the associated graded ring.

Another natural approach is to approximate the group $\pi_p(X)$ by its *nilpotent quotients*, $\pi_p/I^q\pi_p$ (for $q \geq 1$), or by the *associated graded* module over $\mathrm{gr}_I^\bullet \mathbb{Z}\pi_1$, $\mathrm{gr}_I^\bullet \pi_p := \oplus_{q\geq 0}(I^q\pi_p/I^{q+1}\pi_p)$.

Theorem (Dimca-Papadima) When p is maximal, i.e. $p = rank(\mathcal{A}) - 1$, then $\mathrm{gr}_I^i \pi_p$ are combinatorially determined finitely generated abelian groups.

In general, $\mathrm{gr}_I^\bullet \mathbb{Z}\pi_1$ and $\mathrm{gr}_I^0 \pi_p$ are combinatorially determined.

Question: What if we drop the assumption on p?

Theorem (M., Matei, Papadima)The second graded component $\mathrm{gr}_I^1 \pi_p = I\pi_p / I^2\pi_p$ is combinatorially determined and given by an explicit presentation.

Next, we prepare to give some equivalent conditions to the existence of torsion on $\mathrm{gr}_I^1 \pi_p$.

Let $\Lambda^\bullet := \Lambda^\bullet(\mathcal{A})$ be the exterior algebra over $\mathbb{Z}$ generated by the set of hyperplanes of an arbitrary arrangement $\mathcal{A}$.

Let $\mathcal{I}^\bullet := \mathcal{I}^\bullet(\mathcal{A}) \subseteq \Lambda^\bullet$ be the Orlik-Solomon ideal of $\mathcal{A}$, and denote by $\mathsf{A}^\bullet(\mathcal{A}) = \Lambda/\mathcal{I}$ the Orlik-Solomon algebra over $\mathbb{Z}$, known to be torsion-free.

A well known result of Orlik and Solomon states that the $\mathbb{K}$–specialization $\mathsf{A}^\bullet(\mathcal{A})_{\mathbb{K}}$ is isomorphic to the $\mathbb{K}$–cohomology ring of the affine complement of $\mathcal{A}$, for every commutative ring $\mathbb{K}$.

Let $\Lambda^+\mathcal{I} \subseteq \mathcal{I}$ be the decomposable Orlik-Solomon ideal.

We introduce $\mathsf{A}^\bullet_+(\mathcal{A}) := \Lambda/\Lambda^+\mathcal{I}$, the decomposable Orlik-Solomon algebra.

Is $\mathsf{A}^\bullet_+(\mathcal{A})$ also torsion-free?

Theorem(M., Matei, Papadima) Let $\mathcal{A}$ be a hypersolvable and not supersolvable arrangement, and p the π_1-connectivity order. Then the following are equivalent:

1. The second graded piece, $\mathrm{gr}^1_I \pi_p(X)$, has no torsion.
2. The decomposable Orlik-Solomon algebra, $\mathrm{A}^\bullet_+(\mathcal{A})$, is free in degree $\bullet = p + 2$.
3. The graded abelian group of indecomposable OS–relations, $(\mathcal{I}/\Lambda^+\mathcal{I})^\bullet$ is free in degree $\bullet = p + 2$.

Remark When $\mathcal{I}$ is generated in degree 2, the OS-algebra is called *quadratic*, then $\mathrm{A}^\bullet_+(\mathcal{A})$ has no torsion.

Definition A *graphic arrangement* is a subarrangement of a braid arrangement. Graphic arrangements can be described in terms of finite simple graphs. Denote by $\mathcal{A}_\Gamma$ the arrangement associated to the graph Γ.

Corollary (M., Matei, Papadima) Let $\mathcal{A}$ be a hypersolvable and not supersolvable graphic arrangement. Then $\mathrm{A}_+^\bullet(\mathcal{A})$ is torsion free and the second graded piece, $\mathrm{gr}_I^1 \pi_p(X)$, is a finitely generated free abelian group, with rank explicitly computable from the graph Γ.

Examples of arrangements with p non-maximal that fit our description are easy to give, considering the characterisations of supersolvability, respectively hypersolvability in terms of the graph, for graphic arrangements.