Quantum symmetry in homological representations of braid groups and applications

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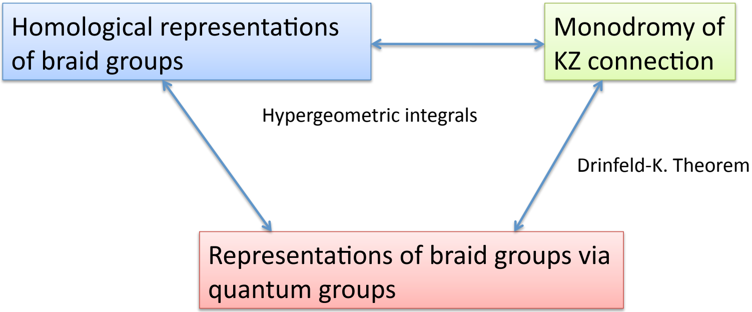
Quantum symmetry in representations of braid groups



Plan

- Homological representations
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- Homological representations and dual Garside structures
- Categorification of KZ connection

Configuration spaces

$\mathcal{F}_n(X)$: configuration space of ordered distinct n points in X.

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\},$$

$$\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$$

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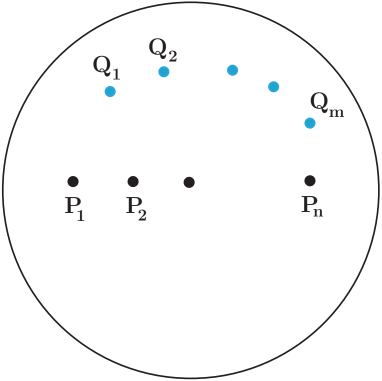
Suppose $X = D$ (two dimensional disc).

$$\pi_1(\mathcal{F}_n(X)) = P_n, \quad \pi_1(\mathcal{C}_n(X)) = B_n$$

Relative configuration spaces

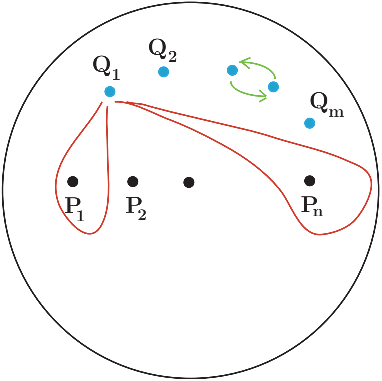
Fix $P = \{(1,0), \cdots, (n,0)\} \subset D$. $\Sigma = D \setminus P$

$$\mathcal{F}_{n,m}(D) = \mathcal{F}_m(\Sigma), \quad \mathcal{C}_{n,m}(D) = \mathcal{F}_m(\Sigma)/\mathfrak{S}_m$$



Homology of relative configuration spaces

$$H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z}$$



Abelian coverings

Consider the homomorphism

$$\alpha : H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$.

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defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$.
Composing with the abelianization map $\pi_1(\mathcal{C}_{n,m}(D), x_0) \to H_1(\mathcal{C}_{n,m}(D); \mathbf{Z})$, we obtain the homomorphism

$$\beta : \pi_1(\mathcal{C}_{n,m}(D), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}.$$

$\pi : \widetilde{\mathcal{C}}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$: the covering corresponding to $\mathrm{Ker}\,\beta$.

Homological representations

$H_*(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ considered to be a $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$-module by deck transformations.

Express $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ as the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$.

$$H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$$

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$$H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$$

$H_{n,m}$ is a free R-module of rank

$$d_{n,m} = \begin{pmatrix} m+n-2 \\ m \end{pmatrix}.$$

$B_n \longrightarrow \mathrm{Aut}_R\, H_{n,m}$: LKB representations $(m > 1)$

KZ connections

$\mathfrak{g}$: complex semi-simple Lie algebra.
$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \leq i \leq n$ representations.

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Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over the configuration space $X_n = \mathcal{F}_n(\mathbf{C})$ with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\omega \wedge \omega = 0$$

Monodromy representations of braid groups

As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

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We shall express the horizontal sections of the KZ connection : $d\varphi = \omega\varphi$ in terms of homology with coefficients in local system homology on the fiber of the projection map

$$\pi : X_{m+n} \longrightarrow X_n.$$

$X_{n,m}$: fiber of π, $\quad Y_{n,m} = X_{n,m}/\mathfrak{S}_m$

Representations of $sl_2(\mathbf{C})$

$\mathfrak{g} = sl_2(\mathbf{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\lambda \in \mathbf{C}$
M_λ : Verma module of $sl_2(\mathbf{C})$ with highest weight vector v such that

$$Hv = \lambda v, \ \ Ev = 0$$

M_λ is spanned by

$$v, Fv, F^2v, \cdots$$

Spece of null vectors

$\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n, \quad |\Lambda| = \lambda_1 + \cdots + \lambda_n$
Consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$.

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m : non-negative integer

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (|\Lambda| - 2m)x\}$$

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The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$, hence it acts on the space of null vectors $N[|\Lambda| - 2m]$.
The monodromy of KZ connection

$$\theta_{\kappa,\lambda} : P_n \longrightarrow \mathrm{Aut}\, N[|\Lambda| - 2m]$$

Comparison theorem

We fix a complex number λ and consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$.

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

Theorem

There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the homological representation $\rho_{n,m}$ with the specialization

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

Local system over the configuration space

$\pi : X_{n+m} \to X_n$: projection defined by
$(z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n)$.
$X_{n,m}$: fiber of π.

$$\Phi = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1 \le i \le m, 1 \le \ell \le n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \times \prod_{1 \le i < j \le m} (t_i - t_j)^{\frac{2}{\kappa}}$$

(multi-valued function on X_{n+m}).
Consider the local system $\mathcal{L}$ associated with Φ.

Solutions to KZ equation

Notation:
$W[|\Lambda| - 2m]$ has a basis

$$F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots F^{j_n} v_{\lambda_n}$$

with $|J| = j_1 + \cdots + j_n = m$ and $v_{\lambda_j} \in \mathcal{M}_{\lambda_j}$ the highest weight vector.

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Theorem (Schechtman-Varchenko...)

The hypergeometric integral

$$\sum_{|J|=m} \left(\int_\Delta \Phi R_J(z,t) dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

lies in $N[|\Lambda| - 2m]$ and is a solution of the KZ equation, where Δ is a cycle in $H_m(Y_{n,m}, \mathcal{L}^*)$.

Homology basis

For generic λ, κ,

$$H_j(Y_{n,m}, \mathcal{L}^*) \cong 0, \quad j \neq m$$

and we have an isomorphism

$$H_m(Y_{n,m}, \mathcal{L}^*) \cong H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$$

(homology with locally finite chains)

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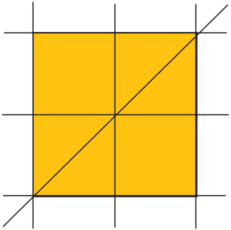
The above homology is spanned by bounded chambers.



bounded chambers : basis of twisted homology
(the case $n = 3, m = 2$).

Homology basis (continued)

For non-negative integers $m_1, \cdots, m_{n-1}$ satisfying

$$m_1 + \cdots + m_{n-1} = m$$

we define a bounded chamber $\Delta_{m_1, \cdots, m_{n-1}}$ in $\mathbf{R}^m$ by

$$\begin{aligned} & 1 < t_1 < \cdots < t_{m_1} < 2 \\ & 2 < t_{m_1+1} < \cdots < t_{m_1+m_2} < 3 \\ & \cdots \\ & n-1 < t_{m_1+\cdots+m_{n-2}+1} + \cdots + t_m < n. \end{aligned}$$

Put $M = (m_1, \cdots, m_{n-1})$ and write Δ_M for $\Delta_{m_1, \cdots, m_{n-1}}$.
The bounded chamber Δ_M defines a homology class $[\Delta_M] \in H_m^{lf}(X_{n,m}, \mathcal{L})$ and its image $\overline{\Delta}_M = \pi_{n,m}(\Delta_M)$ defines a homology class $[\overline{\Delta}_M] \in H_m^{lf}(Y_{n,m}, \mathcal{L})$.
Under a genericity condition $[\overline{\Delta}_M]$ form a basis of $H_m^{lf}(Y_{n,m}, \mathcal{L})$.

Outline of proof of comparison theorem

Now the fundamental solutions of the KZ equation with values in $N[n\lambda - 2m]$ is give by the matrix of the form

$$\left(\int_{\widetilde{\Delta}_M} \omega_{M'} \right)_{M,M'}$$

with $M = (m_1, \cdots, m_{n-1})$ and $M' = (m'_1, \cdots, m'_{n-1})$ such that $m_1 + \cdots + m_{n-1} = m$ and $m'_1 + \cdots + m'_{n-1} = m$. with $\omega_{M'}$ a multivalued m-form on $X_{n,m}$.
The column vectors of the above matrix form a basis of the solutions of the KZ equation with values in $N[n\lambda - 2m]$. Thus the representation $r_{n,m} : B_n \to \mathrm{Aut}\, H_m(Y_{n,m}, \mathcal{L}^*)$ is equivalent to the action of B_n on the solutions of the KZ equation with values in $N[n\lambda - 2m]$.

Theorem

There is an isomorphism

$$N_h[\lambda n - 2m] \cong H_m(Y_{n,m}, \mathcal{L}^*)$$

which is equivariant with respect to the action of the braid group B_n*, where* $N_h[\lambda n - 2m]$ *is the space of null vectors for the corresponding* $U_h(\mathfrak{g})$*-module with* $h = 1/\kappa$*.*

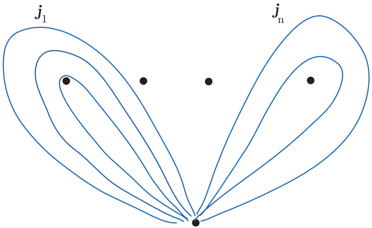
Quantum symmetry for twisted chains

There is the following correspondence:

twisted multi-chains $\Longleftrightarrow$ weight vectors $F^{j_1}v_1 \otimes \cdots \otimes F^{j_n}v_n$

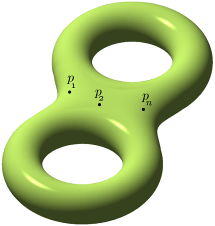
twisted boundary operator $\Longleftrightarrow$ the action of E

$$H_m(Y_{n,m}, \mathcal{L}^*) \Longleftrightarrow N_h[\lambda n - 2m]$$



twisted multi-chains

Conformal Field Theory



$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points
$\lambda_1, \cdots, \lambda_n$: level K highest weights

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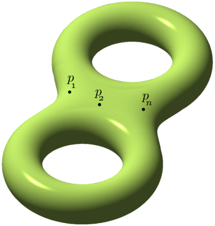

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$\mathcal{H}_\Sigma(p, \lambda)$: **space of conformal blocks**
vector space spanned by holomorphic parts of the WZW partition function.

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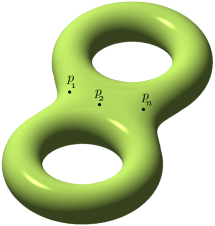

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vector space spanned by holomorphic parts of the WZW partition function.
Geometry : vector bundle over the moduli space of Riemann surfaces with n marked points with projectively flat connection.

Representations of an affine Lie algebra

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$: affine Lie algebra with commutation relation

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \mathrm{Res}_{\xi=0}\, df\, g\ \langle X, Y \rangle c$$

K a positive integer (level)
$\widehat{\mathfrak{g}} = \mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_-$
c acts as $K \cdot \mathrm{id}$.

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c acts as $K \cdot \mathrm{id}$.

λ : an integer with $0 \leq \lambda \leq K$
$\mathcal{H}_\lambda$: irreducible quotient of $\mathcal{M}_\lambda$ called the integrable highest weight modules.

G : the Lie group $SL(2, \mathbf{C})$
$LG = \mathrm{Map}(S^1, G)$: loop group
$\mathcal{L} \longrightarrow LG$: complex line bundle with $c_1(\mathcal{L}) = K$

Geometric background

G : the Lie group $SL(2, \mathbf{C})$
$LG = \mathrm{Map}(S^1, G)$: loop group
$\mathcal{L} \longrightarrow LG$: complex line bundle with $c_1(\mathcal{L}) = K$

The affine Lie algebra $\widehat{\mathfrak{g}}$ acts on the space of sections $\Gamma(\mathcal{L})$.
The integrable highest weight modules $\mathcal{H}_\lambda$, $0 \leq \lambda \leq K$, appears as sub representations.
As the infinitesimal version of the action of the central extension of $\mathrm{Diff}(S^1)$ the Virasoro Lie algebra acts on $\mathcal{H}_\lambda$.

Suppose $0 \leq \lambda_1, \cdots, \lambda_n \leq K$.
$p_1, \cdots, p_n \in \Sigma$
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_j$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

The space of conformal blocks - definition -

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The space of conformal blocks is defined as

$$\mathcal{H}_\Sigma(p, \lambda) = (\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n})/(\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathfrak{g} \otimes \mathcal{M}_p$ acts diagonally via Laurent expansions at $p_1, \cdots, p_n$.

Conformal block bundle

Σ_g : Riemann surface of genus g
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for any complex structures on Σ_g forms a vector bundle on $\mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus g with n marked points.

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This vector bundle is called the conformal block bundle and is equipped with a natural projectively flat connection. The holonomy representation of the mapping class group is called the quantum representation.

Relation to the space of conformal blocks (g=0)

$\mathcal{H}(p, \lambda)$ is identified with a quotient space of $N[\lambda_{n+1}]$ and there is a map

$$\rho : \mathcal{H}(p, \lambda) \to H^m(\Omega^*(Y_{n,m}), \nabla).$$

so that the map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \to \mathcal{H}(p, \lambda)^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \rho(w)$$

is surjective with $\kappa = K + 2$.

Relation to the space of conformal blocks (continued)

Consider the natural map

$$\alpha : H_m(Y_{n,m}, \mathcal{L}^*) \to H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$$

and put $\mathrm{Im}(\alpha) = H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$ (the set of regularizable cycles).

Relation to the space of conformal blocks (continued)

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and put $\mathrm{Im}(\alpha) = H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$ (the set of regularizable cycles).

Theorem

ϕ induces an isomorphism

$$H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg} \cong \mathcal{H}(p, \lambda)^*$$

equivariant under the action of braids.

Fusion rule

In the case $n = 2$ there is an isomorphism.

$$H_m^{lf}(Y_{2,m}, \mathcal{L}^*)_{reg} \cong \mathcal{H}(p_1, p_2, p_3; \lambda_1, \lambda_2, \lambda_3)^*.$$

The above homology group $H_m^{lf}(Y_{2,m}, \mathcal{L}^*)_{reg}$ is isomorphic to $\mathbf{C}$ if the quantum Clebsch-Gordan condition

$$|\lambda_1 - \lambda_2| \leq \lambda_3 \leq \lambda_1 + \lambda_2$$
$$\lambda_1 + \lambda_2 + \lambda_3 \in 2\mathbf{Z}$$
$$\lambda_1 + \lambda_2 + \lambda_3 \leq 2K$$

is satisfied and is isomorphic to 0 otherwise.

Gauss-Manin connection

$\mathcal{L}$: rank 1 local system over $Y_{n,m}$
$m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1})$
$\mathcal{H}_{n,m}$: local system over X_n with fiber $H_m(Y_{n,m}, \mathcal{L}^*)$

Theorem

There is surjective bundle map to the conformal block bundle

$$\mathcal{H}_{n,m} \longrightarrow \bigcup \mathcal{H}^*_{\mathbf{C}P^1}(p, \lambda)$$

via hypergeometric integrals. The KZ connection is interpreted as Gauss-Manin connection.

cf. Looijenga's work

Dual Garside structure

The bounded chamber basis Δ_M plays an important role in detecting the dual Garside structure from the homological representation with respect to this basis.

Theorem (T. Ito and B. Wiest)

The dual Garside length of a braid word β with respect to the Birman-Ko-Lee band generators is expressed as

$$\textit{max degree}_q\ \rho_{n,m}(\beta) - \textit{min degree}_q\ \rho_{n,m}(\beta).$$

Categorification of KZ connections

There is a work in progress to construct 2-holonomy of KZ connection for braid cobordism based on the 2-connection investigated by L. Cirio and J. Martins of the form

$$A = \sum_{i<j} \omega_{ij}\Omega_{ij}$$

$$B = \sum_{i<j<k} (\omega_{ij} \wedge \omega_{ik} \ P_{jik} + \omega_{ij} \wedge \omega_{jk} \ P_{ijk}),$$

where A has values in the algebra of 2-chord diagrams, a categorification of the algebra of horizontal chord diagrams and

$$\partial B = dA + \frac{1}{2} A \wedge A.$$