Quantum symmetry in homological representations of braid groups and applications

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Quantum symmetry in representations of braid groups

Homological representations of braid groups

Hypergeometric integrals

Monodromy of KZ connection

Drinfeld-K. Theorem

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Homological representations
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Relation to KZ connection (comparison theorem)
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- Space of conformal blocks and Gauss-Manin connection
- Homological representations and dual Garside structures
- Categorification of KZ connection
\( \mathcal{F}_n(X) : \) configuration space of ordered distinct \( n \) points in \( X \).

\[
\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\},
\]

\[
\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n
\]
Configuration spaces

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\[
\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n
\]

Suppose \( X = D \) (two dimensional disc).

\[
\pi_1(\mathcal{F}_n(X)) = P_n, \quad \pi_1(\mathcal{C}_n(X)) = B_n
\]
Fix $P = \{(1, 0), \cdots, (n, 0)\} \subset D$. $\Sigma = D \setminus P$

\[ F_{n,m}(D) = F_m(\Sigma), \quad C_{n,m}(D) = F_m(\Sigma)/\mathfrak{S}_m \]
$H_1(C_{n,m}(D); \mathbb{Z}) \cong \mathbb{Z}^\oplus n \oplus \mathbb{Z}$
Consider the homomorphism

\[ \alpha : H_1(C_{n,m}(D); \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \]

defined by \( \alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y) \).
Consider the homomorphism

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defined by \( \alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y) \).

Composing with the abelianization map

\[ \pi_1(C_{n,m}(D), x_0) \rightarrow \text{H}_1(C_{n,m}(D); \mathbb{Z}) \],

we obtain the homomorphism

\[ \beta : \pi_1(C_{n,m}(D), x_0) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}. \]

\[ \pi : \tilde{C}_{n,m}(D) \rightarrow C_{n,m}(D) : \text{the covering corresponding to Ker } \beta. \]
Homological representations

\[ H_\ast(\tilde{C}_{n,m}(D); \mathbb{Z}) \] considered to be a \( \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \)-module by deck transformations.

Express \( \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \) as the ring of Laurent polynomials \( R = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}] \).

\[ H_{n,m} = H_m(\tilde{C}_{n,m}(D); \mathbb{Z}) \]
$H_*(\tilde{C}_{n,m}(D); \mathbb{Z})$ considered to be a $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$-module by deck transformations.

Express $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$ as the ring of Laurent polynomials $R = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$.

$$H_{n,m} = H_m(\tilde{C}_{n,m}(D); \mathbb{Z})$$

$H_{n,m}$ is a free $R$-module of rank

$$d_{n,m} = \binom{m + n - 2}{m}.$$

$B_n \rightarrow \text{Aut}_R H_{n,m} : \text{LKB representations } (m > 1)$
\( g \): complex semi-simple Lie algebra.
\( \{I_\mu\} \): orthonormal basis of \( g \) w.r.t. Killing form.
\( \Omega = \sum_\mu I_\mu \otimes I_\mu \)
\( r_i : g \to \text{End}(V_i) \), \( 1 \leq i \leq n \) representations.
$\mathfrak{g}$ : complex semi-simple Lie algebra.
$\{I_\mu\} : \text{orthonormal basis of } \mathfrak{g} \text{ w.r.t. Killing form.}$

$\Omega = \sum_\mu I_\mu \otimes I_\mu$

$r_i : \mathfrak{g} \rightarrow \text{End}(V_i), \, 1 \leq i \leq n \text{ representations.}$

$\Omega_{ij} : \text{the action of } \Omega \text{ on the } i\text{-th and } j\text{-th components of } V_1 \otimes \cdots \otimes V_n.$

$$\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa \in \mathbb{C} \setminus \{0\}$$

$\omega$ defines a flat connection for a trivial vector bundle over the configuration space $X_n = \mathcal{F}_n(\mathbb{C})$ with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\omega \wedge \omega = 0$$
As the holonomy we have representations

$$\theta_\kappa : P_n \rightarrow GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \rightarrow GL(V^\otimes n).$$
As the holonomy we have representations

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We shall express the horizontal sections of the KZ connection : \( d\varphi = \omega \varphi \) in terms of homology with coefficients in local system homology on the fiber of the projection map

\[ \pi : X_{m+n} \longrightarrow X_n. \]

\( X_{n,m} : \text{fiber of } \pi, \quad Y_{n,m} = X_{n,m}/\mathfrak{S}_m \)
Representations of $sl_2(\mathbb{C})$

$g = sl_2(\mathbb{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\lambda \in \mathbb{C}$

$M_\lambda$: Verma module of $sl_2(\mathbb{C})$ with highest weight vector $v$ such that

$$Hv = \lambda v, \quad Ev = 0$$

$M_\lambda$ is spanned by

$$v, Fv, F^2v, \ldots$$
\[ \Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n, \quad |\Lambda| = \lambda_1 + \cdots + \lambda_n \]

Consider the tensor product \( M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \).
Spece of null vectors

\[ \Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n, \quad |\Lambda| = \lambda_1 + \cdots + \lambda_n \]

Consider the tensor product \( M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \).

\( m \) : non-negative integer

\[ W[|\Lambda| - 2m] = \{ x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} ; \ Hx = (|\Lambda| - 2m)x \} \]
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$$W[\vert \Lambda \vert - 2m] = \{ x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} ; \; Hx = (\vert \Lambda \vert - 2m)x \}$$

The space of null vectors is defined by

$$N[\vert \Lambda \vert - 2m] = \{ x \in W[\vert \Lambda \vert - 2m] ; \; Ex = 0 \}.$$
The space of null vectors

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The KZ connection \( \omega \) commutes with the diagonal action of \( g \) on \( M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \), hence it acts on the space of null vectors \( N[|\Lambda| - 2m] \).

The monodromy of KZ connection

\[ \theta_{k,\lambda} : P_n \longrightarrow \text{Aut} \ N[|\Lambda| - 2m] \]
We fix a complex number $\lambda$ and consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$.

$$N[n\lambda - 2m] \subset M_\lambda^\otimes n.$$

**Theorem**

There exists an open dense subset $U$ in $(\mathbb{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the homological representation $\rho_{n,m}$ with the specialization

$$q = e^{-2\pi \sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi \sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^\otimes n.$$
Local system over the configuration space

\[ \pi : X_{n+m} \to X_n : \text{projection defined by} \]
\[ (z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n). \]
\[ X_{n,m} : \text{fiber of } \pi. \]

\[ \Phi = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1 \leq i \leq m, 1 \leq \ell \leq n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \times \prod_{1 \leq i < j \leq m} (t_i - t_j)^{\frac{2}{\kappa}} \]

(multi-valued function on \( X_{n+m} \)).
Consider the local system \( \mathcal{L} \) associated with \( \Phi \).
Solutions to KZ equation

Notation:
$W[|\Lambda| - 2m]$ has a basis

$$F^Jv = F^{j_1}v_{\lambda_1} \otimes \cdots F^{j_n}v_{\lambda_n}$$

with $|J| = j_1 + \cdots + j_n = m$ and $v_{\lambda_j} \in \mathcal{M}_{\lambda_j}$ the highest weight vector.
Solutions to KZ equation

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Theorem (Schechtman-Varchenko...)

*The hypergeometric integral*

$$\sum_{|J|=m} \left( \int_{\Delta} \Phi R_J(z, t) dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

lies in $N[|\Lambda| - 2m]$ and is a solution of the KZ equation, where $\Delta$ is a cycle in $H_m(Y_{n,m}, \mathcal{L}^*)$. 

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Quantum symmetry in homological representations
Homology basis

For generic $\lambda, \kappa$,

$$H_j(Y_{n,m}, L^*) \cong 0, \quad j \neq m$$

and we have an isomorphism

$$H_m(Y_{n,m}, L^*) \cong H^l_{m}(Y_{n,m}, L^*)$$

(homology with locally finite chains)
For generic $\lambda$, $\kappa$,

$$H_j(Y_{n,m}, L^*) \cong 0, \quad j \neq m$$

and we have an isomorphism

$$H_m(Y_{n,m}, L^*) \cong H_{m}^{lf}(Y_{n,m}, L^*)$$

(homology with locally finite chains)

The above homology is spanned by bounded chambers.

bounded chambers : basis of twisted homology

(the case $n = 3, m = 2$).
Homology basis (continued)

For non-negative integers $m_1, \cdots, m_{n-1}$ satisfying

$$m_1 + \cdots + m_{n-1} = m$$

we define a bounded chamber $\Delta_{m_1, \cdots, m_{n-1}}$ in $\mathbb{R}^m$ by

$$1 < t_1 < \cdots < t_{m_1} < 2$$
$$2 < t_{m_1+1} < \cdots < t_{m_1+m_2} < 3$$
$$\cdots$$
$$n - 1 < t_{m_1+\cdots+m_{n-2}+1} + \cdots + t_m < n.$$ 

Put $M = (m_1, \cdots, m_{n-1})$ and write $\Delta_M$ for $\Delta_{m_1, \cdots, m_{n-1}}$. The bounded chamber $\Delta_M$ defines a homology class $[\Delta_M] \in H^l_m(X_{n,m}, \mathcal{L})$ and its image $\overline{\Delta_M} = \pi_{n,m}(\Delta_M)$ defines a homology class $[\overline{\Delta_M}] \in H^l_m(Y_{n,m}, \mathcal{L})$. Under a genericity condition $[\overline{\Delta_M}]$ form a basis of $H^l_m(Y_{n,m}, \mathcal{L})$. 
Now the fundamental solutions of the KZ equation with values in \( N[n\lambda - 2m] \) is give by the matrix of the form

\[
\begin{pmatrix}
\int & \omega_{M'} \\
\tilde{\Delta}_{M} &
\end{pmatrix}
\]

with \( M = (m_1, \cdots, m_{n-1}) \) and \( M' = (m'_1, \cdots, m'_{n-1}) \) such that 
\( m_1 + \cdots + m_{n-1} = m \) and \( m'_1 + \cdots + m'_{n-1} = m \). with \( \omega_{M'} \) a multivalued \( m \)-form on \( X_{n,m} \).

The column vectors of the above matrix form a basis of the solutions of the KZ equation with values in \( N[n\lambda - 2m] \). Thus the representation \( r_{n,m} : B_n \to \text{Aut} \ H_m(Y_{n,m}, \mathcal{L}^*) \) is equivalent to the action of \( B_n \) on the solutions of the KZ equation with values in \( N[n\lambda - 2m] \).
Quantum symmetry

**Theorem**

There is an isomorphism

\[ N_h[\lambda n - 2m] \cong H_m(Y_{n,m}, \mathcal{L}^*) \]

which is equivariant with respect to the action of the braid group \( B_n \), where \( N_h[\lambda n - 2m] \) is the space of null vectors for the corresponding \( U_h(\mathfrak{g}) \)-module with \( h = 1/\kappa \).
Quantum symmetry for twisted chains

There is the following correspondence:

twisted multi-chains $\iff$ weight vectors $F^{j_1}v_1 \otimes \cdots \otimes F^{j_n}v_n$

twisted boundary operator $\iff$ the action of $E$

$$H_m(Y_{n,m}, \mathcal{L}^*) \iff N_h[\lambda n - 2m]$$
Conformal Field Theory

\((\Sigma, p_1, \cdots, p_n)\) : Riemann surface with marked points

\(\lambda_1, \cdots, \lambda_n\) : level \(K\) highest weights
Wess-Zumino-Witten model

Conformal Field Theory

\((\Sigma, p_1, \cdots, p_n)\) : Riemann surface with marked points
\(\lambda_1, \cdots, \lambda_n : \) level \(K\) highest weights
\(\mathcal{H}_\Sigma(p, \lambda) : \text{space of conformal blocks}\)

vector space spanned by holomorphic parts of the WZW partition function.
(Σ, p₁, · · · , pₙ) : Riemann surface with marked points
λ₁, · · · , λₙ : level $K$ highest weights
$\mathcal{H}_Σ(p, λ)$ : space of conformal blocks
vector space spanned by holomorphic parts of the WZW partition function.
Geometry : vector bundle over the moduli space of Riemann surfaces with $n$ marked points with projectively flat connection.
\[ \hat{g} = g \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c : \text{affine Lie algebra} \text{ with commutation relation} \]

\[ [X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \text{Res}_{\xi=0} df g \langle X, Y \rangle c \]

\( K \) a positive integer (level)
\[ \hat{g} = \mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_- \]
\( c \) acts as \( K \cdot \text{id.} \).
\[ \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c : \text{affine Lie algebra} \text{ with commutation relation} \]

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\( \lambda : \text{an integer with } 0 \leq \lambda \leq K \)
\( \mathcal{H}_\lambda : \text{irreducible quotient of } \mathcal{M}_\lambda \text{ called the integrable highest weight modules.} \)
Geometric background

$G$: the Lie group $SL(2, \mathbb{C})$

$L_G = \text{Map}(S^1, G)$: loop group

$L \to LG$: complex line bundle with $c_1(L) = K$

The affine Lie algebra $\hat{\mathfrak{g}}$ acts on the space of sections $\Gamma(L)$.

The integrable highest weight modules $H_\lambda$, $0 \leq \lambda \leq K$, appear as sub-representations.

As the infinitesimal version of the action of the central extension of $\text{Diff}(S^1)$, the Virasoro Lie algebra acts on $H_\lambda$. 

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Quantum symmetry in homological representations
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$L \to LG$: complex line bundle with $c_1(L) = K$

The affine Lie algebra $\hat{\mathfrak{g}}$ acts on the space of sections $\Gamma(L)$.
The integrable highest weight modules $\mathcal{H}_\lambda$, $0 \leq \lambda \leq K$, appears as sub representations.

As the infinitesimal version of the action of the central extension of $	ext{Diff}(S^1)$ the Virasoro Lie algebra acts on $\mathcal{H}_\lambda$. 
Suppose $0 \leq \lambda_1, \cdots, \lambda_n \leq K$.

$p_1, \cdots, p_n \in \Sigma$

Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.

$\mathcal{H}_j$ : irreducible representations of $\hat{\mathfrak{g}}$ with highest weight $\lambda_j$ at level $K$. 
Suppose \( 0 \leq \lambda_1, \cdots, \lambda_n \leq K \).

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Assign highest weights \( \lambda_1, \cdots, \lambda_n \) to \( p_1, \cdots, p_n \).

\( \mathcal{H}_j \) : irreducible representations of \( \hat{\mathfrak{g}} \) with highest weight \( \lambda_j \) at level \( K \).

\( \mathcal{M}_p \) denotes the set of meromorphic functions on \( \Sigma \) with poles at most at \( p_1, \cdots, p_n \).
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Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
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$\mathcal{M}_p$ denotes the set of meromorphic functions on $\Sigma$ with poles at most at $p_1, \cdots, p_n$.

The space of conformal blocks is defined as

$$\mathcal{H}_\Sigma(p, \lambda) = (\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n})/(\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathfrak{g} \otimes \mathcal{M}_p$ acts diagonally via Laurent expansions at $p_1, \cdots, p_n$.
Conformal block bundle

\[ \Sigma_g : \text{Riemann surface of genus } g \]
\[ p_1, \cdots, p_n : \text{marked points on } \Sigma_g \]
Fix the highest weights \( \lambda_1, \cdots, \lambda_n \).
Conformal block bundle

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Fix the highest weights \( \lambda_1, \cdots, \lambda_n \).

The union

\[ \bigcup_{p_1, \cdots, p_n} \mathcal{H}_{\Sigma_g}(p, \lambda) \]

for any complex structures on \( \Sigma_g \) forms a vector bundle on \( \mathcal{M}_{g,n} \),
the moduli space of Riemann surfaces of genus \( g \) with \( n \) marked points.
Σ

\text{g} \: \text{Riemann surface of genus } g \\
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\text{for any complex structures on } \Sigma_g \text{ forms a vector bundle on } \mathcal{M}_{g,n}, \\
\text{the moduli space of Riemann surfaces of genus } g \text{ with } n \text{ marked points.}

\text{This vector bundle is called the } \text{conformal block bundle} \text{ and is } \\
\text{equipped with a natural } \text{projectively flat connection}. \text{ The} \\
\text{holonomy representation of the mapping class group is called the} \\
\text{quantum representation.}
$\mathcal{H}(p, \lambda)$ is identified with a quotient space of $N[\lambda_{n+1}]$ and there is a map

$$\rho : \mathcal{H}(p, \lambda) \rightarrow H^m(\Omega^*(Y_{n,m}), \nabla).$$

so that the map

$$\phi : H_m(Y_{n,m}, L^*) \rightarrow \mathcal{H}(p, \lambda)^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \rho(w)$$

is surjective with $\kappa = K + 2$. 

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Consider the natural map
\[ \alpha : H_m(Y_{n,m}, \mathcal{L}^*) \to H_{mf}^l(Y_{n,m}, \mathcal{L}^*) \]
and put \( \text{Im}(\alpha) = H_m^l(Y_{n,m}, \mathcal{L}^*)_{\text{reg}} \) (the set of regularizable cycles).
Consider the natural map

\[ \alpha : H_m(Y_{n,m}, L^*) \rightarrow H_{m}^l(Y_{n,m}, L^*) \]

and put \( \text{Im}(\alpha) = H_{m}^l(Y_{n,m}, L^*)_{\text{reg}} \) (the set of regularizable cycles).

**Theorem**

\( \phi \) induces an isomorphism

\[ H_{m}^l(Y_{n,m}, L^*)_{\text{reg}} \cong \mathcal{H}(p, \lambda)^* \]

equivariant under the action of braids.
Fusion rule

In the case $n = 2$ there is an isomorphism.

$$H_{m}^{lf}(Y_{2,m}, \mathcal{L}^{*})_{reg} \cong \mathcal{H}(p_{1}, p_{2}, p_{3}; \lambda_{1}, \lambda_{2}, \lambda_{3})^{*}.$$ 

The above homology group $H_{m}^{lf}(Y_{2,m}, \mathcal{L}^{*})_{reg}$ is isomorphic to $\mathbb{C}$ if the quantum Clebsch-Gordan condition

$$|\lambda_{1} - \lambda_{2}| \leq \lambda_{3} \leq \lambda_{1} + \lambda_{2}$$

$$\lambda_{1} + \lambda_{2} + \lambda_{3} \in 2\mathbb{Z}$$

$$\lambda_{1} + \lambda_{2} + \lambda_{3} \leq 2K$$

is satisfied and is isomorphic to 0 otherwise.
\( \mathcal{L} : \) rank 1 local system over \( Y_{n,m} \)

\[
m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1})
\]

\( \mathcal{H}_{n,m} : \) local system over \( X_n \) with fiber \( H_m(Y_{n,m}, \mathcal{L}^*) \)

**Theorem**

There is surjective bundle map to the conformal block bundle

\[
\mathcal{H}_{n,m} \longrightarrow \bigcup \mathcal{H}^*_{\mathbb{C}P^1}(p, \lambda)
\]

via hypergeometric integrals. The KZ connection is interpreted as Gauss-Manin connection.

cf. Looijenga’s work
The bounded chamber basis $\Delta_M$ plays an important role in detecting the dual Garside structure from the homological representation with respect to this basis.

**Theorem (T. Ito and B. Wiest)**

The dual Garside length of a braid word $\beta$ with respect to the Birman-Ko-Lee band generators is expressed as

$$\max \text{ degree}_q \rho_{n,m}(\beta) - \min \text{ degree}_q \rho_{n,m}(\beta).$$
There is a work in progress to construct 2-holonomy of KZ connection for braid cobordism based on the 2-connection investigated by L. Cirio and J. Martins of the form

\[ A = \sum_{i<j} \omega_{ij} \Omega_{ij} \]

\[ B = \sum_{i<j<k} (\omega_{ij} \wedge \omega_{ik} P_{jik} + \omega_{ij} \wedge \omega_{jk} P_{ijk}), \]

where \( A \) has values in the algebra of 2-chord diagrams, a categorification of the algebra of horizontal chord diagrams and

\[ \partial B = dA + \frac{1}{2} A \wedge A. \]