A Lefschetz hyperplane theorem with an assigned base point

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Part 1. Application.

"If the sum of the Milnor numbers at the singular points of $V(h)$ is large,

then $V(h)$ cannot have a point of large multiplicity, unless $V(h)$ is a cone."

Notations:

- $h \in \mathbb{C}[z_0, \dots, z_n]$ is a homogeneous polynomial of degree d.
- $V(h) := \{h = 0\} \subseteq \mathbb{P}^n$ is the projective hypersurface defined by h.
- $D(h) := \{h \neq 0\} \subseteq \mathbb{P}^n$ is the smooth affine variety defined by h.

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- The *gradient map* of h is the rational map

$$\text{grad}(h) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \qquad z \longmapsto \left(\frac{\partial h}{\partial z_0} : \cdots : \frac{\partial h}{\partial z_n} \right).$$

- The *polar degree* of h is the degree of grad(h).

If $V(h)$ has only isolated singular points, then

$$\deg\bigl(\mathrm{grad}(h)\bigr) = (d-1)^n - \sum_{p \in V(h)} \mu^{(n)}(p),$$

where $\mu^{(n)}(p)$ is the Milnor number of $V(h)$ at p.

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Theorem (A)

Suppose $V(h)$ has only isolated singular points, and let m be the multiplicity of $V(h)$ at one of its points x. Then

$$\deg\big(\text{grad}(h)\big) \geq (m-1)^{n-1},$$

unless $V(h)$ is a cone with the apex x.

Theorem (A)

Suppose $V(h)$ has only isolated singular points, and let m be the multiplicity of $V(h)$ at one of its points x. Then

$$\textit{deg}\big(\textit{grad}(h)\big) \geq (m-1)^{n-1},$$

unless $V(h)$ is a cone with the apex x.

It is interesting to observe how badly the inequality fails when $V(h)$ is a cone over a smooth hypersurface in $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$.

In this case, the polar degree is zero, but the apex of the cone has multiplicity d.

Theorem (B)

Suppose $V(h)$ *has only isolated singular points, and let* $\mu^{(n-1)}$ *be the* $(n-1)$*-th sectional Milnor number of* $V(h)$ *at one of its points* x. *Then*

$$\textit{deg}\big(\textit{grad}(h)\big) \geq \mu^{(n-1)},$$

unless $V(h)$ *is a cone with the apex* x.

Theorem (B)

Suppose $V(h)$ has only isolated singular points, and let $\mu^{(n-1)}$ be the $(n-1)$-th sectional Milnor number of $V(h)$ at one of its points x. Then

$$\text{deg}\big(\text{grad}(h)\big) \geq \mu^{(n-1)},$$

unless $V(h)$ is a cone with the apex x.

A theorem of Teissier says that, locally at any point x,

$$\frac{\mu^{(n)}}{\mu^{(n-1)}} \geq \frac{\mu^{(n-1)}}{\mu^{(n-2)}} \geq \cdots \geq \frac{\mu^{(i)}}{\mu^{(i-1)}} \geq \cdots \geq \frac{\mu^{(1)}}{\mu^{(0)}}.$$

Therefore Theorem (B) implies Theorem (A).

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The inequality of Theorem (B) is tight relative to the degree and the dimension:

For each $d \geq 3$ and $n \geq 2$, there is a degree d hypersurface in $\mathbb{P}^n$ with one singular point, for which the equality holds in Theorem (B).

Conjecture (Dimca and Papadima '03)

A projective hypersurface with only isolated singular points has polar degree 1

if and only if it is one of the following, after a linear change of coordinates:

- $(n \geq 2, d = 2)$ *a smooth quadric*

$$h = z_0^2 + \cdots + z_n^2 = 0.$$

- $(n = 2, d = 3)$ *the union of three nonconcurrent lines*

$$h = z_0 z_1 z_2 = 0.$$

- $(n = 2, d = 3)$ *the union of a smooth conic and one of its tangent*

$$h = z_0(z_1^2 + z_0 z_2) = 0.$$

Theorem (C)

The conjecture of Dimca and Papadima is true.

I will sketch an argument for the above theorems when $n \geq 3$,

using the *Lefschetz hyperplane theorem with an assigned base point*.

Interestingly, our proof does not work for plane curves.

For $n = 2$, one has to argue separately.

(In this case the above statements are theorems of Dolgachev).

Part 2. Lefschetz theorem with an assigned base point

"We may assign a base point when applying Lefschetz hyperplane theorem

(unless our variety has a special geometry with respect to the base point)."

"This extra freedom enables us to relate local and global invariants of the variety."

We drop the assumption that $V(h)$ has only isolated singularities.

Hamm's Lefschetz theory shows that, if H is a general hyperplane in $\mathbb{P}^n$, then

$$\pi_i\Big(D(h), D(h) \cap H\Big) = 0 \quad \text{for} \quad i < n.$$

We refine this result by allowing hyperplanes to have an assigned base point.

Theorem (D)

If H_x is a general hyperplane passing through a point x in $\mathbb{P}^n$, then

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Since $D(h)$ and $D(h) \cap H_x$ are homotopic to CW-complexes of dimensions n and $n-1$ respectively, the vanishing of the homotopy groups implies

$$H_i\Big(D(h), D(h) \cap H_x\Big) = 0 \quad \text{for} \quad i \neq n.$$

An example showing that the first condition is necessary for the conclusion:

Example

Let $V(h)$ be the plane curve consisting of a nonsingular conic containing x, the tangent line to the conic at x, and a general line passing through x. Then

$$H_1\big(D(h), D(h)\cap H_x\big) \simeq H_1\big(S^1\times S^1, S^1\big) \simeq \mathbb{Z}\neq 0.$$

How do we prove something like Theorem (D)?

We go back to the idea of Lefschetz.

Say X is a smooth projective variety of dimension n, and let A be a general codimension 2 linear subspace of a fixed ambient projective space of X.

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The main conclusion of Lefschetz is the isomorphism

$$H_{i+1}(X, X_c) \simeq H_{i-1}(X_c, X_c \cap A), \qquad i < n-1,$$

where X_c is a general member of the pencil of hyperplane sections of X associated to A.

(By induction, one has the vanishing $H_i(X, X_c) = 0$ for $i < n$).

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Let $\mathscr{P}_A$ be the pencil of hyperplanes associated to A, and let $\widetilde{X}$ be the blowup of X along $X \cap A$.

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Let $\mathscr{P}_A$ be the pencil of hyperplanes associated to A, and let $\widetilde{X}$ be the blowup of X along $X \cap A$.

The point of the genericity is that, for such A, the map

$$p : \widetilde{X} \longrightarrow \mathscr{P}_A \simeq \mathbb{P}^1$$

has only isolated singular points.

This main idea has been refined in the last ninety years.

Here is the current version (which we need).

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- $\mathscr{P}_A$ is the pencil of hyperplanes containing the axis A. We write

$$\pi : Y \setminus A \longrightarrow \mathscr{P}_A$$

for the map sending p to the member of $\mathscr{P}_A$ containing p.

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- $\widetilde{Y}$ is the blow-up of Y along $Y \cap A$. We write

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  for the map which agrees with π on $Y \setminus A$.
- $\mathscr{S}$ is a Whitney stratification of $\widetilde{Y}$ which extends $\mathscr{W}|_{Y \setminus A}$.

Definition

The *singular locus* of p with respect to $\mathscr{S}$ is the following closed subset of $\widetilde{Y}$:

$$\text{Sing}_{\mathscr{S}}\ p := \bigcup_{S \in \mathscr{S}} \text{Sing}\ p|_S.$$

We say that $\mathscr{P}_A$ has only *only isolated singular points* with respect to $\mathscr{S}$ if

$$\dim \text{Sing}_{\mathscr{S}}\ p \leq 0.$$

The singular locus of p is a closed subset of $\widetilde{Y}$ because $\mathscr{S}$ is a Whitney stratification.

Theorem (Lefschetz, Andreotti, Frankel, Hamm, Lê, Deligne, Goresky, MacPherson, Nemethi, Siersma, **Tibăr**)

Let X_c be a general member of the pencil on X. Suppose that

1. *the axis A is not contained in V,*
2. *the rectified homotopical depth of X is $\geq n$ for some $n \geq 2$,*
3. *the pencil $\mathscr{P}_A$ has only isolated singular points with respect to $\mathscr{S}$, and*
4. *the pair $(X_c, X_c \cap A)$ is $(n-2)$-connected.*

Then the pair (X, X_c) is $(n-1)$-connected.

Replace the condition 2 by "$\dim X = n \geq 2$ *and X is a local complete intersection*"

if you don't like the rectified homotopical depth.

We are ready for the *inductive* proof of

Theorem (D)

If H_x is a general hyperplane passing through a point x in $\mathbb{P}^n$, then

$$\pi_i\Big(D(h), D(h) \cap H_x\Big) = 0 \quad \text{for} \quad i < n,$$

unless

1. *one of the components of $V(h)$ is a cone with the apex x, or*
2. *the singular locus of $V(h)$ contains a line passing through x.*

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Let A_x be a general codimension 2 linear subspace of $\mathbb{P}^n$ containing x,

and let $\widetilde{\mathbb{P}}^n$ be the blowup of $\mathbb{P}^n$ along A_x.

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a. to show that the two conditions on $V(h)$ are satisfied by $V(h) \cap H_x$, where H_x is a general member of the pencil $\mathscr{P}_{A_x}$,

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b. to produce a Whitney stratification $\mathscr{S}$ of $\widetilde{\mathbb{P}}^n$ such that

   i. $V(h) \setminus A_x$ is a union of strata,

   ii. the map

$$p : \widetilde{\mathbb{P}}^n \longrightarrow \mathscr{P}_{A_x}$$

   has only isolated singularities with respect to $\mathscr{S}$,

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   and

c. to check for $n = 2$, which is an assertion on the fundamental group of plane curve complements.

Of course, we have to use our conditions on $V(h)$ at some point, since otherwise a,b,c are not possible.

Let V be an irreducible subvariety of positive dimension $k+1$ in $\mathbb{P}^n$.

Lemma (a)

The following conditions are equivalent for a point x in $\mathbb{P}^n$.

1. *V is a cone with the apex x.*
2. *For any point y of V different from x, the line joining x and y is contained in V.*
3. *If E_x is a general codimension k linear subspace in $\mathbb{P}^n$ containing x, then every irreducible component of $V \cap E_x$ is a line containing x.*

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The irreducibility assumption is clearly necessary in order to deduce 3 from 4.

Here is another characterization of cones, in the view point of Lefschetz theory.

Let S be a smooth and irreducible locally closed subset of $\mathbb{P}^n$.

(S will be a stratum of the stratification $\mathscr{S}$.)

Lemma (b)

If A_x is a general codimension 2 linear subspace passing through a point x in $\mathbb{P}^n$, then

$$p_{A_x} : S \setminus A_x \longrightarrow \mathscr{P}_{A_x}$$

has only isolated singular points, unless the closure of S in $\mathbb{P}^n$ is a cone with the apex x.

Suppose that

- no component of $V(h)$ is a cone over a smooth variety with the apex x, and
- the singular locus of $V(h)$ does not contain a line passing through x.

Then we can find a Whitney stratification $\mathscr{W}$ of $\mathbb{P}^n$ such that

- $\{x\}$ is a stratum of $\mathscr{W}$,
- $V(h)$ is a union of strata of $\mathscr{W}$, and
- the closure of a stratum of $\mathscr{W} \setminus \big\{\{x\}\big\}$ is not a cone with the apex x.

$\widetilde{\mathbb{P}}^n$ is a subset of $\mathbb{P}^n \times \mathbb{P}^1$.

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$\widetilde{\mathbb{P}}^n$ is a subset of $\mathbb{P}^n \times \mathbb{P}^1$. We use $\mathscr{W}$ to produce the stratification $\mathscr{S}$ of $\widetilde{\mathbb{P}}^n$.

Lemma (b')

Let $\mathscr{S}$ *be the stratification of* $\widetilde{\mathbb{P}}^n$ *with strata*

(1) $\big(S \times \mathbb{P}^1\big) \cap \big(\widetilde{\mathbb{P}}^n \setminus A \times \mathbb{P}^1\big)$ *for* $S \in \mathscr{W} \setminus \big\{\{x\}\big\}$,

(2) $\big(S \times \mathbb{P}^1\big) \cap \big(A \times \mathbb{P}^1\big)$ *for* $S \in \mathscr{W} \setminus \big\{\{x\}\big\}$,

(3) $\{x\} \times \mathbb{P}^1 \setminus E$, *and*

(4) E,

where E *is the set of points at which one of the strata from (1) and (2) fails to be Whitney regular over* $\{x\} \times \mathbb{P}^1$*. Then, for a sufficiently general* A *through* x,

1. $\mathscr{S}$ *is a Whitney stratification, and*
2. p *has only isolated singular points with respect to* $\mathscr{S}$.

Now the base case of the induction.

Let C be a curve in $\mathbb{P}^2$, and x be a point of $\mathbb{P}^2$.

Lemma (c)

Suppose that no line containing x is a component of the curve C. Then for a sufficiently general line L_x passing through x, there is an epimorphism

$$\pi_1\big(L_x \setminus C\big) \longrightarrow \pi_1\big(\mathbb{P}^2 \setminus C\big)$$

induced by the inclusion.

A plane curve may contain a line through x without being a cone, but such a curve cannot be a general hyperplane section through x.

Part 3. Geography of singularities

We use our Lefschetz theorem to justify

Theorem (B)

Suppose $V(h)$ has only isolated singular points, and let $\mu^{(n-1)}$ be the $(n-1)$-th sectional Milnor number of $V(h)$ at one of its points x. Then

$$\textit{deg}\big(\textit{grad}(h)\big) \geq \mu^{(n-1)},$$

unless $V(h)$ is a cone with the apex x.

Proof of Theorem (B) when $n \geq 3$.

We know that

$$\chi\big(D(h)\big) = (-1)^n \text{deg}\big(\text{grad}(h)\big) + \sum_{i=0}^{n-1} (-1)^i (d-1)^i.$$

If $V(h)$ is not a cone with the apex x, choose a general hyperplane H_x containing x

so that

(i) $V(h) \cap H_x$ is smooth outside x, and

(ii) the Milnor number of $V(h) \cap H_x$ at x is the sectional Milnor number $\mu^{(n-1)}$.

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(ii) the Milnor number of $V(h) \cap H_x$ at x is the sectional Milnor number $\mu^{(n-1)}$.

Then

$$\begin{aligned} \text{rank } H_n\big(D(h), D(h) \cap H_x\big) &= (-1)^n \Big(\chi\big(D(h)\big) - \chi\big(D(h) \cap H_x\big)\Big) \\ &= \text{deg}\big(\text{grad}(h)\big) - \mu^{(n-1)} \geq 0. \end{aligned}$$

□

Therefore, if $\deg\big(\text{grad}(h)\big) = 1$, then $\mu^{(n-1)} = 1$ at all the singular points.

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Lemma (d)

Let $(V, \mathbf{0})$ be the germ of an isolated hypersurface singularity at the origin of $\mathbb{C}^n$.

If $\mu^{(n-1)}$ of the germ is equal to 1, then the singularity is of type A_k for some $k \geq 1$.

(The conjecture of DP follows.)

In conclusion, we have

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I believe

Conjecture

This forbidden region is large if n and d are large. More precisely, for any positive integer k, there is no projective hypersurface of polar degree k with only isolated singular points, for sufficiently large n and d.