Geometric Fox Calculus

Eriko Hironaka
Florida State University

Alba Iulia, June 29, 2013

Objects of study: Out (Fn), Mod(S)

Let F_n be the free group on n generators.

$$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inner}(F_n).$$

A subclass of interest is the mapping class group of an oriented surface S with at least one boundary component

$$\begin{aligned} \mathrm{Mod}(S) &= \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0^+(S) \\ &= \mathrm{Aut}(\pi_1(S))/\mathrm{Inner}(\pi_1(S)). \end{aligned}$$

Two invariants of elements $\phi \in \mathrm{Out}(F_n)$:

- the algebraic dilatation
- the geometric dilatation

(ongoing work with Algom-Kfir and Rafi, and Hadari)

Algebraic and geometric dilatation

Each $\phi \in \text{Out}(F_n)$ defines an invertible linear transformation $\phi_* : \mathbb{R}^n \to \mathbb{R}^n$ defined over the integers.

The *algebraic dilatation* of ϕ is given by

$$\lambda_{\text{alg}}(\phi) \quad = \quad \text{spectral radius of } \phi_*$$

The *geometric dilatation* of ϕ is given by

$$\lambda_{\text{geo}}(\phi) = \sup_{\omega \in F_n} \left(\lim_{k \to \infty} \ \ell(\phi^k(\omega))^{\frac{1}{k}} \right),$$

where ℓ is the word length with respect to a fixed set of generators.

Comparison

We can think of words in F_n as loops on a bouquet of n circles of length one, and ℓ_{geo} as the *geometric length* of ω.

Similarly, we can define the *algebraic length* ℓ_{alg} of a word in F_n as the vector norm of its image in $\mathbb{Z}^n \subset \mathbb{R}^n$. This notion of length is degenerate.

In this way, the algebraic and geometric dilatations both measure growth rates of "lengths" of words in F_n under iterations of the map ϕ.

Since $\ell_{\text{alg}}(\omega) \leq \ell_{\text{geo}}(\omega)$, for all $\omega \in F_n$, we have

Proposition $\lambda_{\text{hom}}(\phi) \leq \lambda_{\text{geo}}(\phi)$

Mapping class theory background

Let S be a compact oriented surface of finite type, and $\phi : S \to S$ a homeomorphism.

Theorem (Nielsen-Thurston) ϕ is either periodic, reducible or pseudo-Anosov.

Theorem (Thurston) In the pseudo-Anosov case the lengths of $\phi^n(\gamma)$ grow exponentially regardless of the choice of metric or nontrivial closed curve γ. The geometric dilatation is also independent of the choices, and equals the growth rate of the induced map on the fundamental group of S.

In this setting, where $\phi \in \mathrm{Out}(F_n)$ is induced by a pseudo-Anosov mapping class, we call $\lambda_{\mathrm{geo}}(\phi)$ ($\lambda_{\mathrm{hom}}(\phi)$), the geometric (homological) dilatation of ϕ.

Properties of dilatations (why they are of interest)

The surface case.

- algebraic dilatations:
  - any algebraic unit can occur
- geometric dilatations:
  - they are special algebraic integers: Perron units,
  - their log equals the Teichmüller length of a geodesic on the moduli space of complex structures on S,
  - their minima for a fixed S are hard to calculate, and
  - their relation with other invariants such as volume of mapping torus is not well understood.

For general elements of $\mathrm{Out}(F_n)$, dilatations need not be units.

Useful Example: The simplest hyperbolic braid monodromy

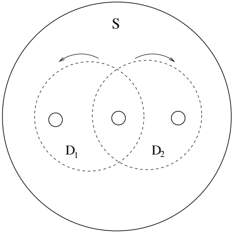

Figure: Mapping class on $S_{0,4}$

The homological dilatation: $\lambda_{\text{hom}} = 1$.

The geometric dilatation: $\lambda_{\text{geo}} = \frac{3+\sqrt{5}}{2} = (\text{golden mean})^2$.
(Use train tracks)

Train tracks

A *train track* τ is a finite 1-complex with smoothings at vertices.

A *folding map* is a surjective train track map that folds a portion of one edge onto an adjacent edge.

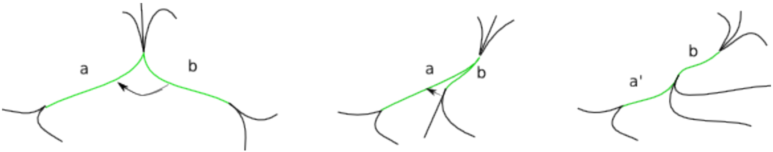

Figure: A folding.

Train track maps and train track automata

A *train track* map $f : \tau \to \tau$ is a composition of folding maps and a homeomorphism that sends a train track τ to itself.

(Bestvina-Handel, Cho-Ham-Los-Song): The directed graph with vertices corresponding to train tracks, solid edges corresponding to folding and dotted edges corresponding to homeomorphisms is called a *train track automaton*.

- Closed loops in train track automata correspond to free group automorphisms.
- If a closed loop has the property that the spectral radius is greater than one, then the free group automorphism is hyperbolic and fully irreducible.

If the train tracks are given a ribbon or fat graph structure, and folding maps preserve this structure, then the loops can be realized as surface homeomorphisms.

Simplest hyperbolic braid monodromy

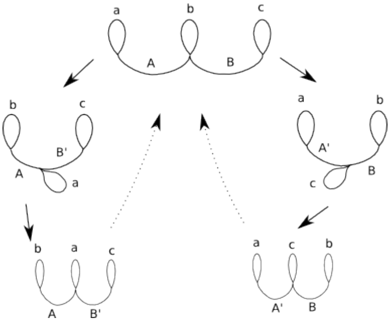

Figure: Train track automaton

Computing dilatation

Theorem (Thurston, Bestvina-Handel, Dowdall-Kapovich-Leininger) If $\phi \in \mathrm{Out}(F_n)$ is fully-irreducible and hyperbolic, then there is a train track τ and a train track map f such that the induced map

$$f_* : \pi_1(\tau) \to \pi_1(\tau)$$

equals ϕ. Furthermore, f is a composition of folding maps, and defines a linear map

$$f : \mathbb{R}^{\mathcal{V}} \to \mathbb{R}^{\mathcal{V}},$$

where $\mathcal{V}$ is the vertex set for τ, and we have

$$\lambda(\phi) = \text{spectral radius } (f_*).$$

Train tracks for the simplest hyperbolic braid monodromy

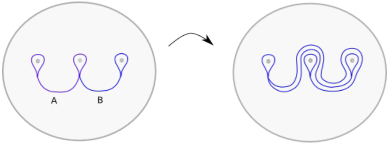

Figure: Train track map

The homological dilatation: $\lambda_{\text{hom}} = 1$.

The geometric dilatation:

$$\lambda_{\text{geo}} = \text{Spec Rad} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{3+\sqrt{5}}{2}$$

Deformations of automorphisms

If we consider the set theoretic union $\bigcup_n \mathrm{Out}(F_n)$, there is a notion of open families of automorphisms.

The Alexander and Teichmüller polynomials package the algebraic and geometric dilatations of automorphisms in natural families.

For pseudo-Anosov surface automorphisms: (Thurston, Fried, Matsumoto, McMullen). The set of monodromies of a hyperbolic 3-manifold M fibered over a circle with the same suspended stable lamination are integer points in an open cone in $H^1(M;\mathbb{R})$. These cones are called *fibered cones* and projectivize via theThurston norm to top dimensional faces (*fibered faces*) of the Thurston norm ball.

Fibered face

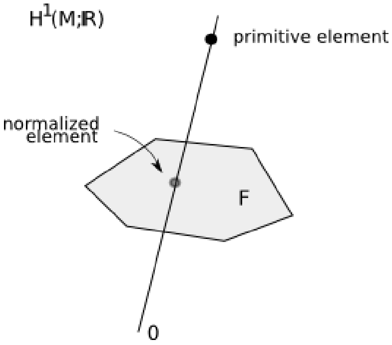

Figure: Primitive element in a fibered face

We have $\mathcal{F}(M) \subset \mathcal{F}(M,F) \subset H^1(M;\mathbb{Z}) \subset H^1(M;\mathbb{R})$.

It follows that rational points on F parameterize elements of $\mathcal{F}(M,F)$.

Example: Deformations of simplest hyperbolic braid

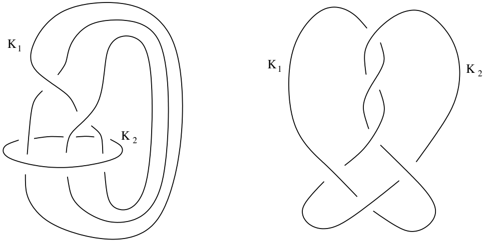

Figure: Mapping torus for simplest hyperbolic braid monodromy.

Fibered face

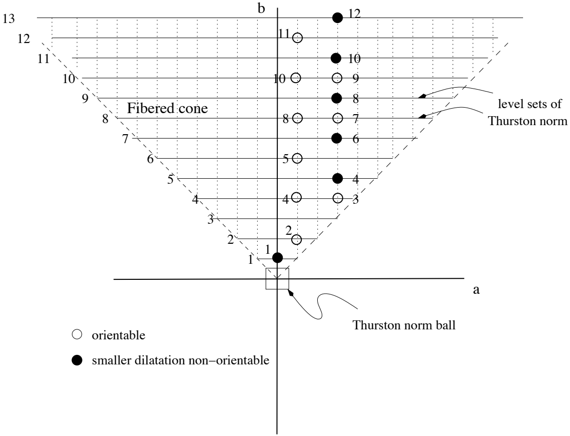

Figure: Fibered face.

Train tracks for deformations

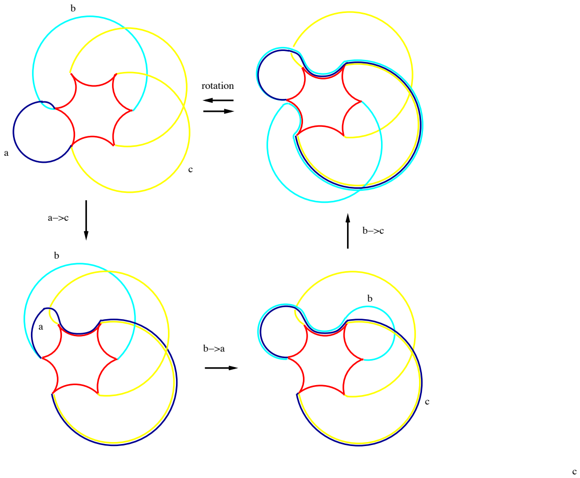

Figure: Train track for $\phi_{(1,2)}$

Train tracks for deformations

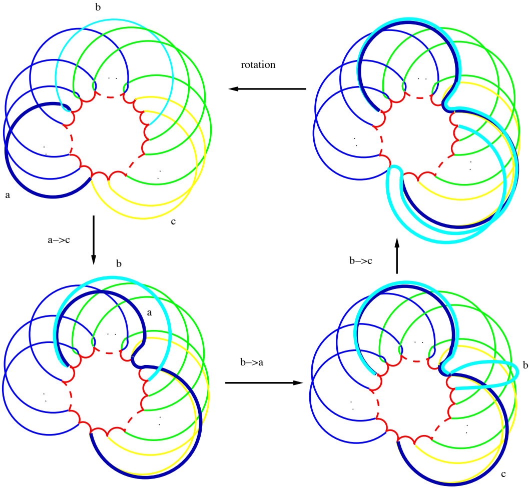

Figure: Train track for $\phi_{(1,n)}$

Deformations of free group automorphisms

A map $\phi \in \mathrm{Out}(F_n)$ is *irreducible* if F_n cannot be decomposed into nontrivial free factors whose conjugacy classes are permuted by ϕ.

ϕ is *fully irreducible* if no power is reducible.

ϕ is *hyperbolic* or *atoroidal* if ϕ^n preserves no conjugacy class of non-trivial element of F_n.

(Dowdall-Kapovich-Leininger) Let $\phi \in \mathrm{Out}(F_n)$ be fully irreducible and hyperbolic, and $\Gamma = F_n \rtimes_\phi \mathbb{Z}$. There is an open cone $V \subset \mathrm{Hom}(G;\mathbb{R})$ so that for all integral $\alpha \in V$, the kernel of α is a finitely generated free group giving Γ another free-by-cyclic structure. Furthemore, the corresponding automorphism ϕ_α is also freely irreducible and hyperbolic.

We call V a *fibered cone neighborhood* of ϕ.

Alexander polynomial: general definition

The Alexander polynomial is an invariant of a finitely presented group Γ.

Let Γ' be the kernel of the abelianization map $\Gamma \to G$. The abelianization of the commutator subgroup has a finite presentation as a ZG module:

$$(\mathbb{Z}G)^r \xrightarrow{A} (\mathbb{Z}G)^s \longrightarrow \Gamma'/\Gamma" \longrightarrow 0.$$

The Alexander polynomial of Γ is the generator of the smallest principal ideal containing the first fitting ideal of this presentation.

Fox calculus gives a way to calculate an $s \times r$ matrix representation for A with $\mathbb{Z}G$ entries given a presentation of Γ. The Alexander polynomial is independent of the presentation.

Fox Calculus

Given a surjective group homomorphism $\rho : F_r \to G$, where G is a free abelian group, define, for $i = 1, \ldots, r$, $D_i : F_r \to \mathbb{Z}G$ so that

(i) $D_i(x_j) = \delta_{i,j} \cdot \mathrm{id}_G$, and

(ii) $D_i(fg) = D_i(f) + \mu(f) D_i(g)$.

It follows that

(iii) $D_i(1) = 0$,

(iv) $D_i(x_i^m) = 1 + \cdots + x_i^{m-1}$, for $m \geq 1$, and

(iv) $D_l(x_i^{-m}) = -x_i^{-1} - \cdots - x_i^{-m}$, for $m \leq -1$.

Interpretation via unbranched coverings

Consider the unbranched covering $X \to Y$, where Y is the bouquet of r circles and the covering is defined by

$$F_r \to \Gamma \to G.$$

One can picture X as an r-dimensional grid. Then the columns of the Alexander matrix A are simply the lifts to X of the words $R_1, \ldots, R_s$ considered as closed paths on Y.

Fox lifts: Example

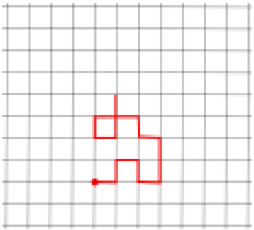

Figure: Fox lift of the word $\omega = xyxy^{-1}xy^2x^{-1}yx^{-2}y^{-1}xy^2$

$$
\begin{aligned}
D_x(\omega) &= 1 + xy + x^2 - x^2y^2 - xy^3 - y^3 + y^2 \\
D_u(\omega) &= x - x^2 + x^3 + x^3y + x^2y^2 - y^2 + xy^2 + xy^3
\end{aligned}
$$

Alexander matrix

Let $F_s \to F_r \to \Gamma$ be a presentation and $\Gamma \to G$ a surjective group homomorphism (G abelian).

Theorem (Fox) Then the *Alexander matrix* is given by

$$A = [D_j(R_i)],$$

where $R_1, \ldots, R_s$ generate the image of a presentation, and the *Alexander polynomial* Δ is a generator of the smallest principal ideal containing the first fitting ideal of A.

Remark: The first fitting ideal of A is the defining ideal for the jumping locus for first homology.

Alexander polynomial: Free-by-cyclic case

We are interested in the case when Γ is "fibered".

Assume $\Gamma = F_n \rtimes_\phi \mathbb{Z}$, for some $\phi \in \mathrm{Out}(F_n)$. Let G be the abelianization of Γ modulo torsion.

Let F_n be generated by $x_1, \ldots, x_n$. Then $G = K \times \langle h \rangle$, where K is the image of F_n in G.

Define $T = [D_j(\phi(x_i))]$, where the Fox derivatives are taken with respect to $F_n \to K$.

Proposition $\Delta = \det(hI - T)$.

Specializations of polynomials

Let G be a finitely generated free abelian group, and $f \in \mathbb{Z}G$:

$$f = \sum_g a_g g.$$

The specialization of f at $\alpha \in \mathrm{Hom}(G;\mathbb{Z})$ is given by

$$f^{(\alpha)}(x) = \sum_g a_g x^{\alpha(g)}.$$

Given a polynomial $f(t) \in \mathbb{Z}[t]$, the *house* is given by

$$|f| = \max\{|\mu| \;:\; f(\mu) = 0\}$$

Algebraic dilatations on a fibered cone neighborhood

Let $V \subset \mathrm{Hom}(\Gamma; \mathbb{Z})$ be a fibered cone neighborhood of a hyperbolic, fully-irreducible automorphism $\phi \in \mathrm{Out}(F_n)$.

Proposition For integral $\alpha \in V$,

$$\lambda_{\text{hom}}(\phi_\alpha) = |\Delta^{(\alpha)}|.$$

Pre-Alexander polynomial.

Let $\phi \in Out(F_n)$, and let G be the abelianization of F_n.

Let $D : F_n \to (\mathbb{Z}G)^n$ be the corresponding Fox derivative, and let Δ be the characteristic polynomial of the matrix with columns given by $D(x_i)$.

We still have

$$\lambda_{\text{hom}}(\phi_\alpha) = |\Delta^\alpha|,$$

for $\alpha \in \text{Hom}(G : \mathbb{Z})$ such that α extends to $\Gamma = F_n \rtimes_\phi \mathbb{Z}$.

This will be useful later.

Theorem (McMullen) Let $\phi \in \text{Mod}(S)$ and let (M, V) be the mapping torus and fibered cone $V \subset H^1(M;\mathbb{R})$. Let $G = H_1(M;\mathbb{Z})$/torsion. There is an element $\Theta \in \mathbb{Z}G$ such that for all integral $\alpha \in V$,

$$\lambda(\phi_\alpha) = |\Theta^{(\alpha)}|.$$

Theorem (Algom–Kfir-H-Rafi) Let $\phi \in \text{Out}(F_n)$ be a fully-irreducible hyperbolic element, and let (Γ, V) be the associated free-by-cyclic group and a fibered cone neighborhood of ϕ in $\text{Hom}(\Gamma;\mathbb{R})$. Then there is an element $\Theta \in \mathbb{Z}G$ such that for all integral $\alpha \in V$,

$$\lambda(\phi_\alpha) = |\Theta^{(\alpha)}|.$$

Fox lifts: Revisited

How to compute Θ?

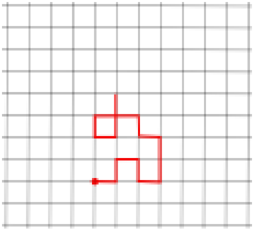

Figure: Fox lift of the word $\omega = xyxy^{-1}xy^2x^{-1}yx^{-2}y^{-1}xy^2$

$$
\begin{aligned}
D_x(\omega) &= 1 + xy + x^2 - x^2y^2 - xy^3 - y^3 + y^2 \\
D_u(\omega) &= x - x^2 + x^3 + x^3y + x^2y^2 - y^2 + xy^2 + xy^3
\end{aligned}
$$

Geometric Fox calculus on train tracks

Let τ be a train track for a hyperbolic, fully-irreducible automorphism $\phi \in Out(F_n)$. In particular, $F_n = \pi_1(\tau)$, so the abelianization map $F_n \to G$ defines an unbranched covering $\rho : \widetilde{\tau} \to \tau$.

The generators $x_1, \ldots, x_n$ of F_n determine closed edge paths on τ. Thus, we can define Fox lifts as before, except that now all edges are counted positively.

We are guaranteed to have no backtracking because of the train track property.

Geometric Fox Calculus

Lifts of train tracks can be done in an analogous way as Fox calculus, but we don't need to worry about orientations of edges.
Let $x_1, \dots, x_n$ be the edges of a train track. Let $F_n = \langle x_1, \dots, x_n \rangle$, $G = \mathbb{Z}^n$, and $\mu : F_n \to G$ the abelianization map. Define operators

$$D_i^+ : F_n \to \mathbb{Z}G,$$

$i = 1, \dots, m$, to be the map satisfying

1. $D_i^+(1) = 0$,
2. $D_i^+(x_j\omega) = \delta_{j,m} + \mu(x_j)D_i^+(\omega)$, for all $\omega \in F_m$,
3. $D_i^+(x_j^{-1}\omega) = \mu(x_j)^{-1} + \mu(x_j^{-1})D_i^+(\omega)$.

The image of D_i^+ lies on $\mathbb{Z}^+G$, that is, they are of the form

$$\sum_{g \in G} a_g g,$$

where $a_g \geq 0$ for all $g \in G$.

Application: Lanneau-Thiffeault Question

Question: Let δ_g^+ be the smallest geometric dilatation achieved by an orientable pseudo-Anosov map. Then is it true that for g even

$$\delta_g^+ = |x^{2g} - x^{g+1} - x^g - x^{g-1} + 1| \ ?$$

A pseudo-Anosov mapping class $\phi \in \mathrm{Mod}(S)$ leaves invariant a pair of transverse measured foliations. The map ϕ is *orientable* if its invariant foliations are orientable, or equivalently

$$\lambda_{\text{hom}}(\phi) = \lambda_{\text{geo}}(\phi).$$

Theorem (H_) There is a sequence (S_g, ϕ_g) of oirentable pseudo-Anosov mapping classes of genus g with $\lambda(\phi_g) = |x^{2g} - x^{g+1} - x^g - x^{g-1} + 1|$ and (S_g, ϕ_g) project to a sequence on a single fibered face converging to (S_s, ϕ_s).

Proof. Look at deformations of the simplest hyperbolic braid monodromy.

Lifting maps for simplest hyperbolic braid

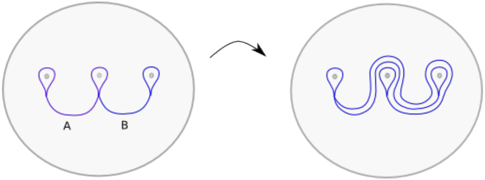

Figure: Train track map for (S_s, ϕ_s)

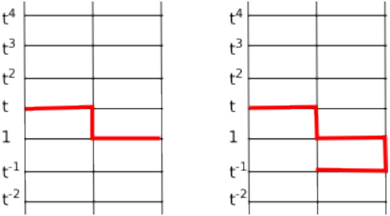

Figure: Lift of $\phi(A)$ and $\phi(B)$

Alexander and Teichmüller polynomial for simplest hyperbolic braid

$\Delta = u^2 - u(1 - t - t^{-1}) + 1$ is the characteristic polynomial of the map

$$\begin{bmatrix} -t & -t \\ 1 & 1 - t^{-1} \end{bmatrix}$$

$\Theta = u^2 - u(1 + t + t^{-1}) + 1$ is the characteristic polynomial of the map

$$\begin{bmatrix} t & t \\ 1 & 1 + t^{-1} \end{bmatrix}$$

Corollary $\phi_{a,b}$ is orientable if and only if a is odd and b is even.

Fibered face

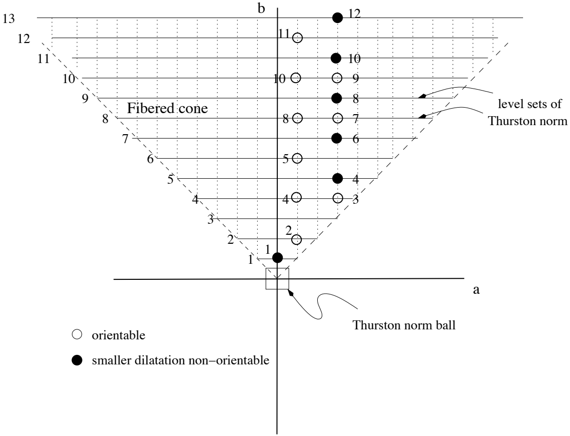

Figure: Fibered face.

New Direction: Weighted growth rates

...with A. Hadari (in progress).

Assume for simplicity that the bouquet B of n-circles is a train track for $\phi \in \mathrm{Out}(F_n)$. (If not replace the bouquet by a train track for ϕ.)

Consider the universal abelian covering $\mathcal{L} \to B$.

For each $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ let $\mathcal{L}_s$ be the deformation of $\mathcal{L}$ so that the lengths of edges emanating in the positive direction from $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ equals $e^{s \cdot a} = e^{s_1 a_1 + \cdots + s_n a_n}$.

Given $\omega \in F_n$, let $\ell_s(\omega)$ be its lift to $\mathcal{L}_s$ based at $0 \in \mathbb{Z}^n$. (Lifting at g corresponds to multiplying the length of ℓ_s by $e^{s \cdot g}$.)

Growth rates

Given a hyperbolic, fully-irreducible $\phi \in Out(F_n)$, and $s \in \mathbb{R}^n$, let

$$\lambda_s(\phi) = \lim_{n \to \infty} \ell_s(\phi^n(\omega))^{\frac{1}{n}},$$

called the weighted growth rate. Then

- $\lambda_s(\phi)$ does not depend on ω,
- $\lambda_0(\phi) = \lambda(\phi)$,
- $\lambda_s(\phi)$ is a convex function on s, and hence has a unique minimum.

Proof uses work of McMullen on characteristic polynomials of digraphs labeled by elements of $\mathbb{Z}G$, where G is a free abelian group. (In this case G is the abelianization of F_n).

Example: train track is bouquet of circles

Consider the free group automorphism

$$\begin{aligned} \phi : F_2 &\rightarrow F_2 \\ x &\mapsto xy \\ y &\mapsto yxy \end{aligned}$$

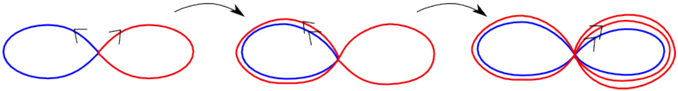

Figure: Folding diagram for ϕ

The bouquet of circles is a train track.

Fox lift

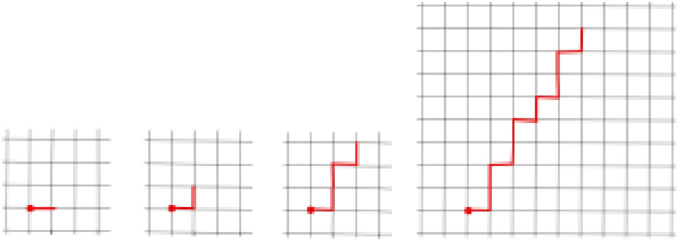

Figure: Fox lift for ϕ

The homological and geometric dilatations of ϕ are given by

$$\text{Spec Rad} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = |x^2 - 3x + 1| = \frac{3+\sqrt{5}}{2}.$$

Fox derivatives

The Fox derivatives of $\phi(x)$ and $\phi(y)$ is given by

$$\begin{aligned} D^+(x) = D(x) &= (1, t_x) \\ D^+(y) = D(y) &= (t_y, 1 + t_x t_y) \end{aligned}$$

and

$$\Delta(u, t_x, t_y) = \Theta(t_x, t_y) = u^2 - u(2 + t_x t_y) - 1.$$

ϕ has no invariant cohomology, so it is isolated, and the only specialization of Δ or Θ corresponding to a fibration is at $t_x = t_y = 1$.

One can check looking at the polynomial Δ that the smallest dilatation is obtained at $a = (0, 0)$.

Torelli case

$\phi \in \mathrm{Out}(F_n)$ is Torelli, if it acts trivially on the abelianization G of F_n.

In this case, all elements $\alpha \in \mathrm{Hom}(G;\mathbb{Z})$ correspond to deformations of ϕ on a fibered face of $\Gamma = F_n \rtimes_\phi \mathbb{Z}$.

Recently, A. Hadari showed that if ϕ is Torelli, the traces of the Fox lifts on $\mathcal{L}$ of words in F_n behave nicely under iterations and normalizations. We believe this phenomena can be understood using Teichmüller matrix theory.

Mulțumesc