Real models of arrangements and polytopes

Giovanni Gaiffi

University of Pisa

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G-, Permutonestohedra. arxiv 1305.6097, May 2013

What is a permutonestohedron

A permutonestohedron is a polytope associated to a root system Φ with finite Coxeter group W.
It can be constructed (in the euclidean vector space V that is spanned by the roots) in the following way:

- we (carefully) place a nestohedron inside the fundamental chamber;
- we consider the W orbit of this nestohedron;
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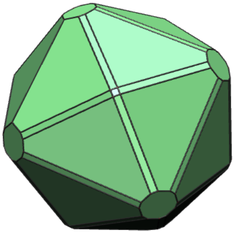
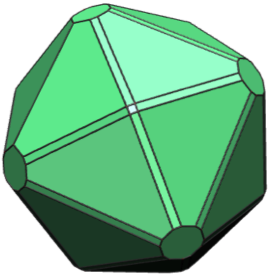
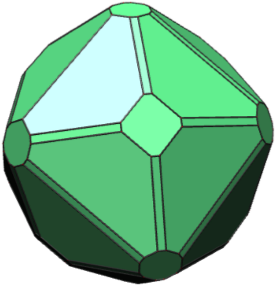
The name permutonestohedra comes from the remark that Kapranov's permutoassociahedra belong to this family:



Example: the 3-dimensional Kapranov's permutoassociahedron is the minimal permutonestohedron of type A_3

There are several permutonestohedra associated with a root system Φ:





The minimal and the maximal permutonestohedron of type A_3

Building sets, nested sets and nestohedra

Definition

A building set of the power set $\mathcal{P}(\{1,2,...,n\})$ *is a subset* $\mathcal{B}$ *of* $\mathcal{P}(\{1,2,...,n\})$ *such that:*

a) *If* $A, B \in \mathcal{B}$ *have nonempty intersection, then* $A \cup B \in \mathcal{B}$.
b) *The set* $\{i\}$ *belongs to* $\mathcal{B}$ *for every* $i \in \{1,2,...,n\}$.
c) *The set* $\{1,2,...,n\}$ *belongs to* $\mathcal{B}$.

Postnikov, Reiner, Williams (Documenta Math. 2008), Postnikov (IMRN 2009).
See also Feichtner and Kozlov (Selecta Math. 2004) and Petric (J. Alg. Comb. 2013).

Definition

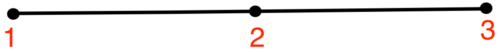
A subset $\mathcal{S}$ of a building set $\mathcal{B}$ is a nested set if and only if the following three conditions hold:

a) *For any $I, J \in \mathcal{S}$ we have that either $I \subset J$ or $J \subset I$ or $I \cap J = \emptyset$.*

b) *Given elements $\{J_1, ..., J_k\}$ ($k \geq 2$) of $\mathcal{S}$ pairwise not comparable with respect to inclusion, their union is not in $\mathcal{B}$.*

c) *$\mathcal{S}$ contains all the sets of $\mathcal{B}$ which are maximal with respect to inclusion.*

The nested set complex $\mathcal{N}(\mathcal{B})$ is the poset of all the nested sets of $\mathcal{B}$ ordered by inclusion.



$$\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$$

$$\{\{1\}, \{3\}, \{1,2,3\}\} \in \mathcal{N}(\mathcal{B}) \qquad \{\{2\}, \{2,3\}, \{1,2,3\}\} \in \mathcal{N}(\mathcal{B})$$

A nestohedron $P_{\mathcal{B}}$ is a polytope whose face poset, ordered by reverse inclusion, is isomorphic to the nested set complex $\mathcal{N}(\mathcal{B})$.

The nestohedron in the example is the 2-dimensional Stasheff's associahedron (a pentagon).

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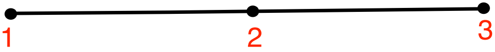

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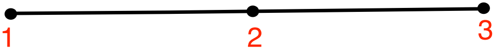

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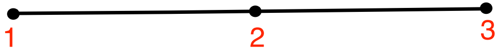

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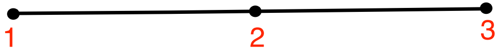

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Connection with De Concini-Procesi models

Let us consider a root system Φ in V with finite Coxeter group W, and a basis of *simple roots* $\Delta = \{\alpha_1, ..., \alpha_n\}$ for Φ.

Let us denote by

- $\mathcal{C}_\Phi$ the building set of all the subspaces that can be generated as the span of some of the roots in Φ.
- $\mathcal{F}_\Phi$ the building set made by all the subspaces which are spanned by the irreducible root subsystems of Φ.

For instance, if $\Phi = A_n$:

$$< \alpha_2, \alpha_3, \alpha_4 + \alpha_5 > \in \mathcal{F}_\Phi \subset \mathcal{C}_\Phi$$

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Let $\mathcal{A}$ be the root hyperplane arrangement given by the hyperplanes orthogonal to the roots in Φ.

Let $\mathcal{M}(\mathcal{A})$ be the complement in V to $\mathcal{A}$. Let $\mathcal{G}$ be one of the two building sets $\mathcal{F}_\Phi$, $\mathcal{C}_\Phi$. [1] There is an open embedding

$$\phi : \mathcal{M}(\mathcal{A})/\mathbb{R}^+ \longrightarrow S(V) \times \prod_{A \in \mathcal{G}} S(A)$$

Definition

We denote by $CY_\mathcal{G}$ the closure of the image of ϕ.

[1] This works for every geometric building set associated to Φ.

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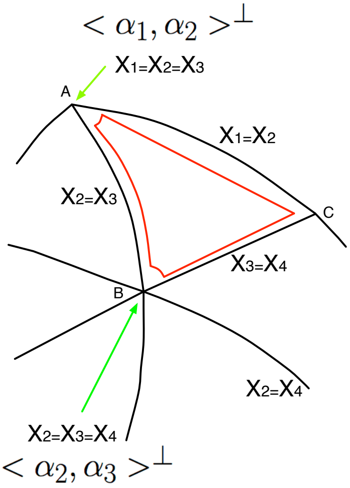
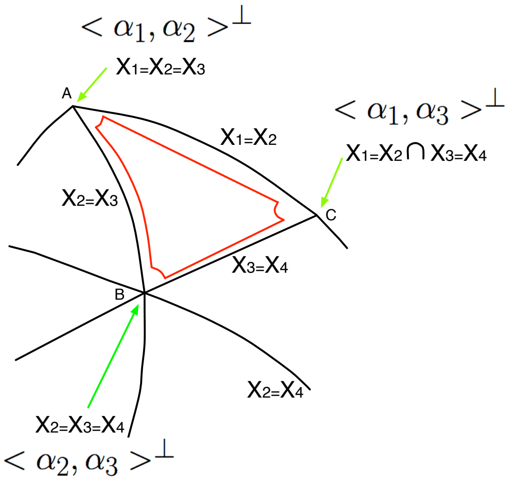
- $CY_{\mathcal{G}}$ is a smooth manifold with corners;
- $CY_{\mathcal{G}}$ has as many connected components as the number of chambers of $\mathcal{M}(\mathcal{A})$;
- its boundary components are in correspondence with the elements of the building set $\mathcal{G}$, and the intersection of some of these boundary components is nonempty if and only if they correspond to a nested set.

Moreover in G-, (IMRN 2003) these connected components were realized inside the chambers, as the complement of a suitable set of tubolar neighbouroods of the subspaces in $\mathcal{G}$, giving rise to (non linear) realizations of nestohedra.



Case A_3, building set of irreducibles $\mathcal{F}_{A_3}$.



Case A_3, maximal building set $\mathcal{C}_{A_3}$.

Our goal

We show how to construct a linear realization of $CY_{\mathcal{G}}$ (that works for any root system ϕ and any W invariant building set $\mathcal{G}$ - all the W-invariant building sets of types A, B, C, D have been classified in G-, Serventi (Eur. J. Comb. 2013))
such that its convex hull is a polytope, the permutonestohedron $P_{\mathcal{G}}(\Phi)$.

- the realization of the nestohedra inside the chambers turns out to be a generalization of Stasheff and Shnider's construction (1997) of the Stasheff's associahedron.
- also Reiner and Ziegler's 'Coxeter associahedra' (Mathematika, 1994) are obtained as convex hulls of Stasheff's associahedra that lie in the chambers.

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Semisum of roots and inequalities

Let $\mathcal{G}$ be one of the two building sets $\mathcal{F}_\Phi$, $\mathcal{C}_\Phi$ defined above.
We will denote by $\mathcal{G}_{fund}$ the intersection of $\mathcal{G}$ with the set of subspaces which are generated by some subset of Δ.
For instance, if $\Phi = A_n$ and $\mathcal{G} = \mathcal{F}_\Phi$:

$$< \alpha_2, \alpha_3, \alpha_4 + \alpha_5 > \notin \mathcal{G}_{fund}$$

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Given $A \in \mathcal{G}_{fund}$, if $A \cap \Phi$ is an irreducible root subsystem, we denote by π_A the semisum of all its positive roots.

$$\pi_A = \frac{1}{2} \sum_{\alpha \in \Phi^+ \cap A} \alpha$$

If $A \cap \Phi$ is not irreducible, and splits into the irrducible subsystems $\Phi_1, ..., \Phi_s$, then $\pi_A = \sum \pi_{\Phi_i}$ where π_{Φ_i} is the semisum of all the positve roots of Φ_i.

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Let us consider two subspaces $B \subset A$ in $\mathcal{G}_{fund}$ of dimension $j < i$ respectively and write π_A and π_B as non negative linear combinations of the simple roots. We denote by a the maximum coefficient of π_A and by b the minimum coefficient of π_B and put $R_B^A = \frac{a}{b}$.

We then define R_j^i as the maximum among all the R_B^A with A, B as above.

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Let B be a subspace in $\mathcal{G}_{fund}$ that can be expressed as a (not redundant) sum of some subspaces $B_1, B_2, ..., B_r$ in $\mathcal{G}_{fund}$ $(r > 1)$. Then we have

$$\sum_{i=1}^{r} R_{dim\ B_i}^{dim\ B} \pi_{B_i} \gtrdot \pi_B$$

where $\alpha \gtrdot \beta$ means that the difference $\alpha - \beta$ can be expressed as a non negative linear combination of the simple roots.

Definition

A list of positive real numbers $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{n-1} < \epsilon_n = a$ is suitable *if, for every set of subspaces $B, B_1, B_2, ..., B_r$ in $\mathcal{G}_{fund}$ as above it satisfies*

$$\epsilon_{dim\ B} > \sum_{i=1}^{r} R_{dim\ B_i}^{dim\ B} \epsilon_{dim\ B_i}$$

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Proposition

A list of positive real numbers $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{n-1} < \epsilon_n = a$ *such that* $\epsilon_i > 2R^i_{i-1}\epsilon_{i-1}$ *for every* $i = 2, ..., n$ *is suitable.*

The defining hyperplanes

Given a suitable list $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{n-1} < \epsilon_n = a$, we put

$$H_V = \{x \in V \mid (x, \pi_V) = a\}$$

and, for every $A \in \mathcal{G}_{fund} - \{V\}$

$$H_A = \{x \in V \mid (x, \pi_V - \pi_A) = a - \epsilon_{dim\ A}\}$$

These are the defining hyperplanes of the component of $CY_{\mathcal{G}}$ (a nestohedron) which lies in the fundamental chamber.
This nestohedron lies in H_V and is bounded by $H_A \cap H_V$.

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Each of these hyperplanes is invariant with respect to the action of a parabolic subgroup; this is the reason why in the global construction, when we consider the convex hull of all the nestohedra which lie in the chambers, the extra facets of the permutonestohedron $P_{\mathcal{G}}(\Phi)$ appear.

Remark

If $\Phi = A^1 \times \cdots \times A^1$ *(the boolean arrangement) then all the numbers* $R_j^i = 1$ *and the condition is* $\epsilon_i > 2\epsilon_{i-1}$*, and our construction, in the irreducible case, coincides with Stasheff and Shnider's construction of the associahedron.*

The pairs coset-nested set

The faces of $P_{\mathcal{G}}(\Phi)$ are in bijective correspondence with the pairs $(\sigma H, \mathcal{S})$, where:

- $\mathcal{S}$ is a nested set of $\mathcal{G}_{fund}$ which contains V and has labels attached to its minimal elements: if A is a minimal element, its label is either the subgroup W_A of W or the trivial subgroup $\{e\}$.
- σH is a coset of W, where H is the subgroup of W given by the direct product of all the labels and $\sigma \in W$.

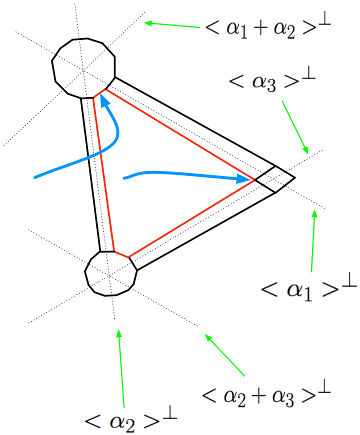


Figure : The blue arrows indicate respectively the vertex $(\{e\}, \{V, < \alpha_1 >, < \alpha_3 >\})$ and the edge $(\{e\}, \{V, < \alpha_1, \alpha_2 >\})$.

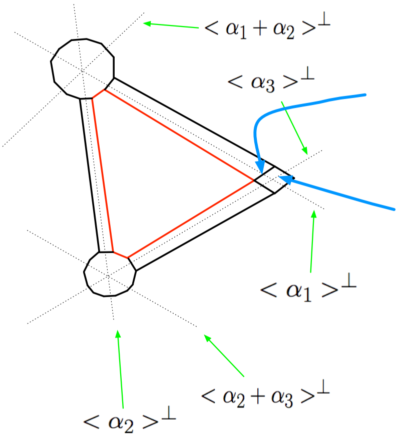


Figure : The blue arrows indicate respectively the edge $(W_{<\alpha_1>}, \{V, \underline{< \alpha_1 >}, < \alpha_3 >\})$ and the facet $(W_{<\alpha_1>} \times W_{<\alpha_3>}, \{V, \underline{< \alpha_1 >}, \underline{< \alpha_3 >}\})$.

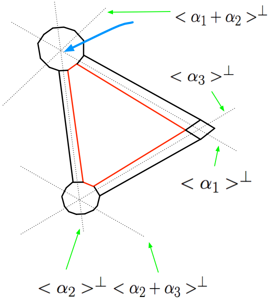


Figure : The blue arrow indicates the facet $(W_{<\alpha_1,\alpha_2>}, \{V, \underline{<\alpha_1, \alpha_2>}\})$.

The dimension of the face $(\sigma H, \mathcal{S})$ is given by $n - |\mathcal{S}| + l$ where l is the number of nontrivial labels.

The extra facets

Theorem

The facet $(\sigma W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}, \{V, \underline{A_1}, \underline{A_2}, \ldots, \underline{A_k}\})$ *of* $P_{\mathcal{G}}(\Phi)$ *is combinatorially equivalent to the product*

$$P_{\overline{\mathcal{G}}} \times P_{\mathcal{G}^{A_1}}(\Phi \cap A_1) \times \cdots \times P_{\mathcal{G}^{A_k}}(\Phi \cap A_k)$$

Here $\overline{\mathcal{G}}$ is the 'quotient' building set in V/D ($D = A_1 \oplus A_2 \oplus \cdots \oplus A_k$) defined by

$$\overline{\mathcal{G}} = \{(C + D)/D \mid C \in \mathcal{G}_{fund}\}$$

Examples of face counting

Theorem

For every $0 \leq k \leq n-2$ *the number of faces of codimension* $k+1$ *of the minimal permutonestohedron* $P_{\mathcal{F}_{A_{n-1}}}(A_{n-1})$ *is*

$$\sum_{\lambda \in \Lambda_n,\ l(\lambda) \geq 2+k} w(\lambda) \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_{l(\lambda)}!} \left[\frac{1}{k+1} \binom{l(\lambda)-2}{k} \binom{l(\lambda)+k}{k} \right]$$

Remark

In the case of the faces of codimension $n-1$*, i.e. the vertices, the formula of Theorem above specializes to* $C_{n-1} n!$ *where* C_{n-1} *is the Catalan number* $\frac{1}{n}\binom{2n-2}{n-1}$.

Theorem

For every $0 \le k \le n-2$ the number of faces of codimension $k+1$ of the maximal permutonestohedron $P_{\mathcal{C}_{A_{n-1}}}(A_{n-1})$ is

$$\sum_{\lambda \in \Lambda_n,\ l(\lambda) \ge 2+k} w(\lambda) \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_{l(\lambda)}!} \left[\sum_{1 < j_1 < \cdots < j_k < l(\lambda) = j_{k+1}} \prod_{t=1}^{k} \binom{j_{i+1}-1}{j_i - 1} \right]$$

Remark

The formula of the Theorem above in particular claims that the vertices of $P_{\mathcal{C}_{A_{n-1}}}(A_{n-1})$ are $(n-1)!n!$, as expected from a 'permutopermutohedron'.

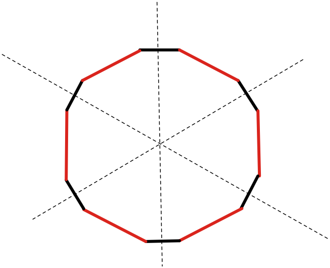
The automorphism group of the permutonestohedron

Let $\mathcal{G}$ be $\mathcal{F}_\Phi$, $\mathcal{C}_\Phi$, or any building set invariant with respect to the group $Aut(\Phi)$ (i.e. the semidirect product of the Weyl group with the automorphism group Γ of the Dynkin diagram).

Theorem

The group $Aut(\Phi)$ is included in $Aut(P_{\mathcal{G}}(\Phi))$.
There are infinite suitable lists $\epsilon_1 < \cdots < \epsilon_n = a$ such that $Aut(\Phi) = Aut(P_{\mathcal{G}}(\Phi))$. More precisely, once a is fixed, for all the possible suitable lists whose greater number is a, except possibly for a finite number, we have $Aut(\Phi) = Aut(P_{\mathcal{G}}(\Phi))$.

The permutonestohedron $P_{\mathcal{F}_{A_2}}(A_2)$ is a dodecagon



Once $a = \epsilon_2$ is fixed, there is only one admissible value for ϵ_1 such that $P_{\mathcal{F}_{A_2}}(A_2)$ is regular.
If it is not regular its automorphism group coincides with $Aut\ A_2 \cong S_3 \rtimes \mathbb{Z}_2$.
If it is regular its automorphism group is the full dihedral group with 24 elements.

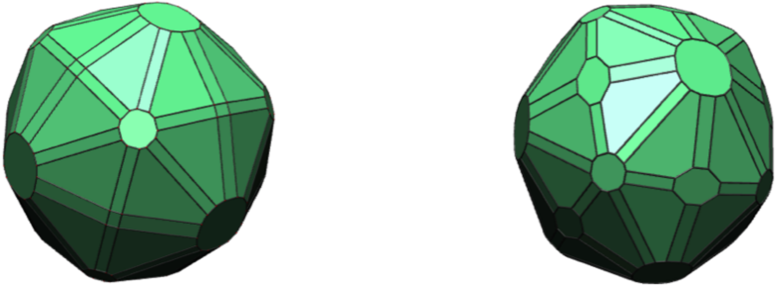


Figure : The minimal (on the left) and the maximal permutonestohedron of type B_3. For any choice of the suitable list, their automorphism group is $Aut(B_3) = W_{B_3} \cong \mathbb{Z}_2^3 \rtimes S_3$

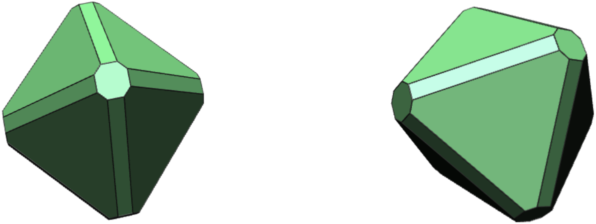


Figure : Two pictures of the maximal nestohedron in the three dimensional boolean case $(A_1 \times A_1 \times A_1)$. Its automorphism group coincides with $Aut\ A_1^3 \cong \mathbb{Z}_2^3 \rtimes S_3$ $(\cong W_{B_3})$.

The extended action in the braid case

This part is a work in progress with F. Callegaro.

There is a well know S_{n+1} action on the De Concini-Procesi model $Y_{\mathcal{F}_{A_{n-1}}}$, that is a quotient of $CY_{\mathcal{F}_{A_{n-1}}}$: it comes from the isomorphism with the moduli space $M_{0,n+1}$.
This action can be lifted to the face poset of $CY_{\mathcal{F}_{A_{n-1}}}$ (not to the full permutonestohedron).

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Let $\Delta = \{\alpha_0, \alpha_1, ..., \alpha_{n-1}\}$ be a basis for the root system of type A_n (we added to A_{n-1} the extra root α_0) and let $\tilde{\Delta}$ be the set of roots of the affine diagram.
Let $\sigma \in S_{n+1}$.

$$\sigma(\{e\}, \{V, A\}) = ?$$

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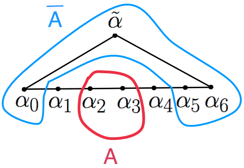
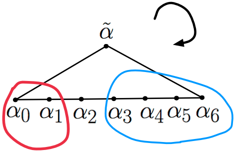
If at least one of the roots in σA, say $\sigma(\alpha_2)$, contains α_0 in its support...





Let $\sigma \in S_{n+1}$.

$$\sigma(\{e\}, \{V, A\}) =?$$

Let $w = \sigma(0, 1, 2, 3, 4, 5, ..., n)^r$ be the representative of the coset $\sigma < (0, 1, 2, 3, 4, 5, ..., n) >$ which fixes 0.

- if some of the roots in σA contain α_0 in their support, then we denote by $\overline{A}$ the subspace generated by all the roots of $\tilde{\Delta}$ which are orthogonal to A and we put $\sigma(\{e\}, \{A\}) = (w\{e\}, \{V, w^{-1}\sigma\overline{A}\})$.
- otherwise $\sigma(\{e\}, \{V, A\}) = (w\{e\}, \{V, w^{-1}\sigma A\})$.

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- This S_{n+1} action on the face poset of the minimal model $CY_{\mathcal{F}_{A_{n-1}}}$ doesn't extend to the face poset of the maximal model.
- Why? The maximal model is too small...
- we can embed its face poset in the face poset of a bigger model, where the S_{n+1} action exists.

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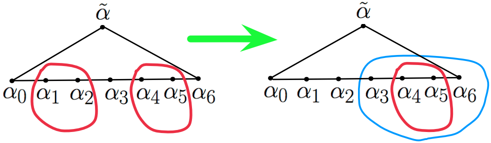
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- Let us consider all the strata in $CY_{\mathcal{F}_{A_{n-1}}}$. They form a building set $\mathcal{B}(n-1)$ in the sense of MacPherson and Procesi (Selecta Math. 1998) and G- (IMRN 2003).
  It is also a combinatorial building set, according to Feichtner and Kozlov's definition of building set of a meet semilattice, see also Petric's paper (J. Alg. Comb. 2013).
- Therefore we can blow up all the strata in this building set and obtain a new model $CY_{\mathcal{B}(n-1)}$ (the linear construction extends too, and we have a wider class of permutonestohedra).
  On the face poset of $CY_{\mathcal{B}(n-1)}$, as well as on $Y_{\mathcal{B}(n-1)}$, there is the S_{n+1} action.

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$$CY_{\mathcal{F}_{A_{n-1}}} \subset CY_{\mathcal{C}_{A_{n-1}}} \subset CY_{\mathcal{B}(n-1)}$$

On the face posets of the left and right models there is the S_{n+1} action.

Theorem (informal claim)

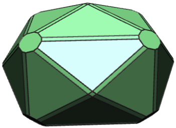
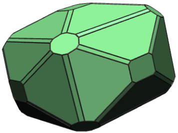
The face poset of $CY_{\mathcal{B}(n-1)}$ is the closure of the face poset of $CY_{\mathcal{C}_{A_{n-1}}}$ under the S_{n+1} action.

Scheme of models and their S_{n+1}-closures in $CY_{\mathcal{B}(n-1)}$:

$$CY_{\mathcal{F}_{A_{n-1}}} \subset \text{intermediate model } \subset CY_{\mathcal{C}_{A_{n-1}}}$$

$$CY_{\mathcal{F}_{A_{n-1}}} \subset \text{intermediate closure} \subset CY_{\mathcal{B}(n-1)}$$

This produces several geometrical realizations of representations of S_{n+1}, in particular of the regular representation and of $Ind_G^{S_{n+1}} Id$, for any subgroup G of the cyclic group $<(0, 1, ..., n)>$.

Thank you for your attention!