Toric arrangements

Emanuele Delucchi
(joint with Giacomo d'Antonio)
Universität Bremen

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Alba Iulia
June 28., 2013.

This talk will offer some snapshots of

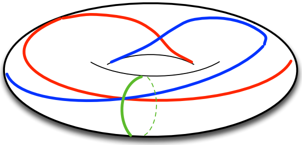
- d'Antonio, D.; *A Salvetti complex for toric arrangements and its fundamental group.* International Mathematics Research Notices (IMRN), 2011.

- d'Antonio, D.; *Minimality of toric arrangements.* To appear in Journal of the European Mathematical Society (JEMS), 2013.

Toric arrangements

A complexified toric arrangement is a set

$$\mathscr{A} = \{(\chi_i, a_i)\}_{i=1,\dots,n} \subseteq \mathbb{Z}^d \times S^1.$$



With

$$\mathbb{Z}^d \simeq \mathrm{Hom}((\mathbb{C}^*)^d, \mathbb{C}^*),$$
$$S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

define

$$K_i = \chi_i^{-1}(a_i) \subseteq (\mathbb{C}^*)^d.$$

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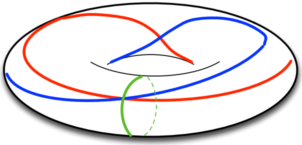

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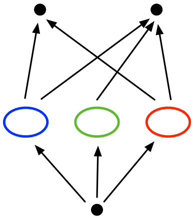
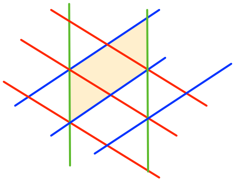
Layers of $\mathscr{A}$: conn. comp. of intersection of some of the K_i.

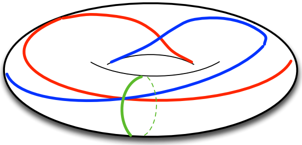
$\mathcal{C}(\mathscr{A})$:= poset of layers ordered by reverse inclusion.



Face structure

$\mathscr{A}^{\upharpoonright}$





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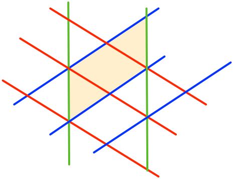

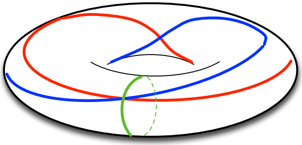

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FACE STRUCTURE

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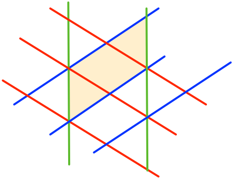

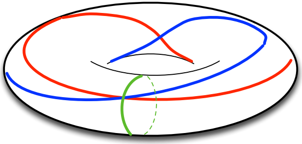

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$$\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}^{\upharpoonleft})/\mathbb{Z}^d,$$

an acyclic category.

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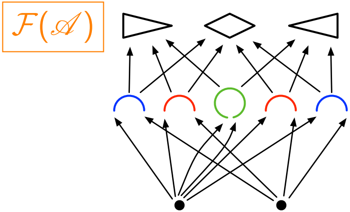

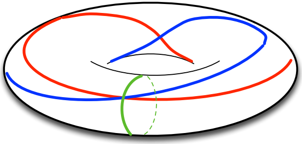

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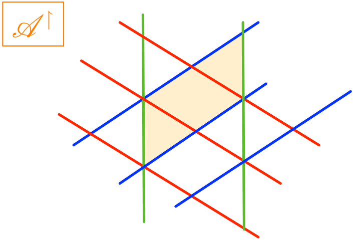
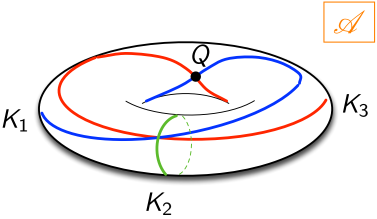
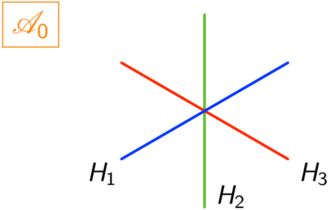
Combinatorial bookkeeping I







$$\mathscr{A}_0 = \{H_1, \ldots, H_n\}$$

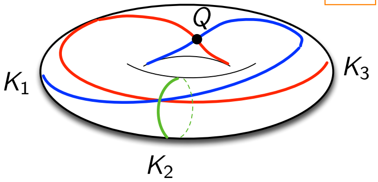
is a central arrangement in $\mathbb{R}^d$, consisting of the translate at the origin of a lift of each K_i.

Combinatorial bookkeeping I

Given $Y \in \mathcal{C}(\mathscr{A})$ let

$$\mathscr{A}[Y] = \{H_i \in \mathscr{A}_0 : Y \subseteq K_i\}.$$

$\mathscr{A}$



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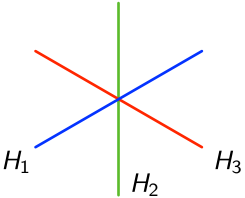
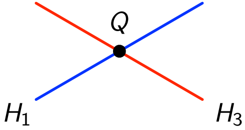


$\mathscr{A}[Q]$



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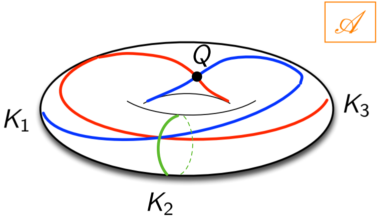

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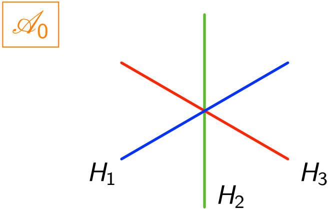
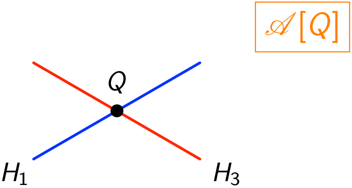
For $i = 1, \ldots, d$, let

$$\mathscr{N}_i := \{(Y, N) \mid Y \in \mathcal{C}(\mathscr{A}), N \in \mathrm{nbc}(\mathscr{A}[Y]), |N| = \mathrm{rk}\,\mathscr{A}[Y] = i\}.$$

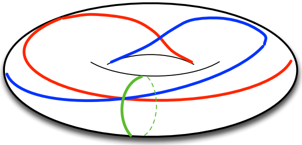




Topology

We consider

$$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \bigcup_{i=1}^{n} K_i.$$



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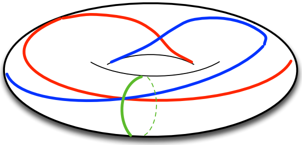
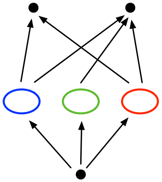
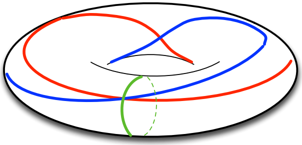




- [Looijenga '98; De Concini, Procesi '05] The Poincaré polynomial of $M(\mathscr{A})$ can be computed in terms of $\mathcal{C}(\mathscr{A})$.
- [dD '11] Presentation of $\pi_1(M(\mathscr{A}))$ in terms of $\mathcal{F}(\mathscr{A})$.

Poincaré polynomial

[De Concini and Procesi, '05]
The Poincaré polynomial of $M(\mathscr{A})$ is



$$P(M(\mathscr{A}), t) = \sum_{j=1}^{d} |\mathscr{N}_j|(1+t)^{d-j}t^j$$

Moreover, when $\mathscr{A}$ is unimodular the multiplicative structure of $H^*(M(\mathscr{A}), \mathbb{C})$ is computed.

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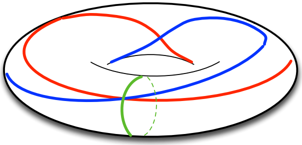

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- Is there torsion in $H^*(M(\mathscr{A}),\mathbb{Z})$?
- What is the multiplicative structure of $H^*(M(\mathscr{A}),\mathbb{Z})$?

- When is $M(\mathscr{A})$ a $K(\pi,1)$?
- Can the category $\mathcal{F}(\mathscr{A})$ be defined axiomatically?

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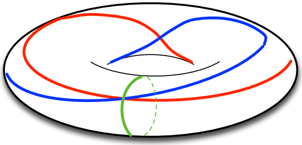

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- Is there torsion in $H^*(M(\mathscr{A}), \mathbb{Z})$? [Today]
- What is the multiplicative structure of $H^*(M(\mathscr{A}), \mathbb{Z})$? [Ongoing project w. F. Callegaro]
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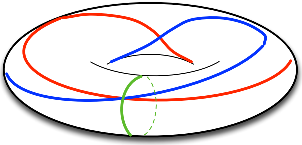

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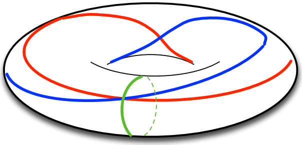

The action on $\mathscr{A}^\upharpoonright$ extends to a cellular action on $\mathsf{Sal}(\mathscr{A}^\upharpoonright)$.

The Salvetti category of $\mathscr{A}$ is the acyclic category

$$\mathsf{Sal}(\mathscr{A}) := \mathsf{Sal}(\mathscr{A}^\upharpoonright)/\mathbb{Z}^d.$$

Theorem [Moci and Settepanella '11, dD '11]. $\mathsf{Sal}(\mathscr{A})$ *can be defined in terms of* $\mathcal{F}(\mathscr{A})$*, and we have a homotopy equivalence*

$$\Delta(\mathsf{Sal}(\mathscr{A})) \simeq M(\mathscr{A}).$$

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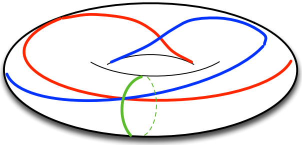


The diagram of acyclic categories

$$\begin{aligned} \mathscr{D} : \mathcal{F}(\mathscr{A}) &\to \mathbf{AC} \\ F &\mapsto \mathrm{Sal}(\mathscr{A}[\|F\|]), \end{aligned}$$

with inclusions as morphisms, is “geometric”.

Theorem [dD ‘12].

$$\operatorname{colim} \mathscr{D} \simeq \mathrm{Sal}(\mathscr{A})$$

Order on chambers

Definition. *Let C_1, C_2 be chambers of $\mathscr{B}$, F any face.*
$S(C_1, C_2) \subset \mathscr{B}$: *the set of hyperplanes separating C_1 from C_2,*

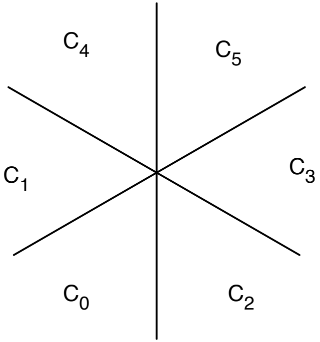
Fix a chamber B.

The partial order

$$C_1 \leq_B C_2 \Leftrightarrow S(B, C_1) \subseteq S(B, C_2)$$

defines the poset of regions $\mathcal{P}_B(\mathscr{B})$ based at B.

Let $\prec$ be a linear extension of $\mathcal{P}_B(\mathscr{B})$.



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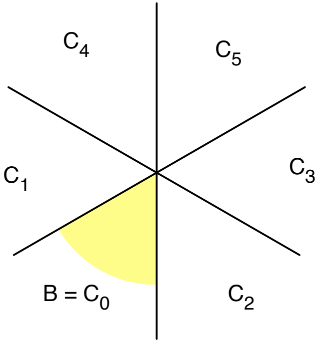

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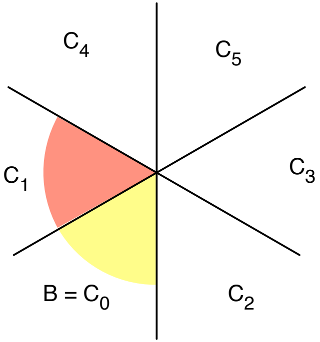

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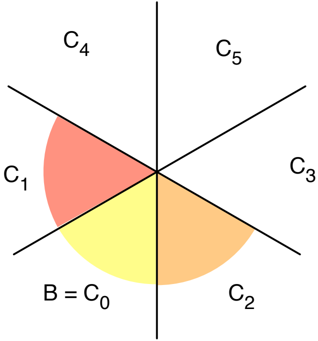

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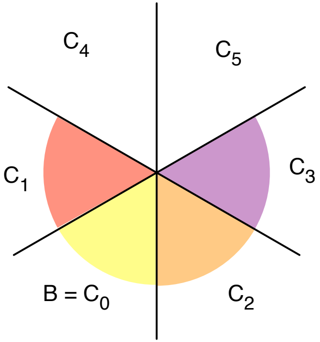

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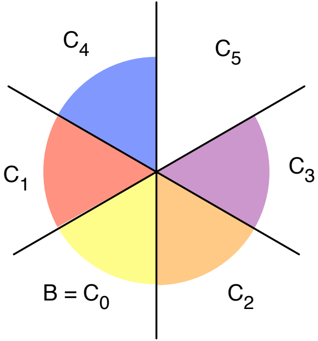

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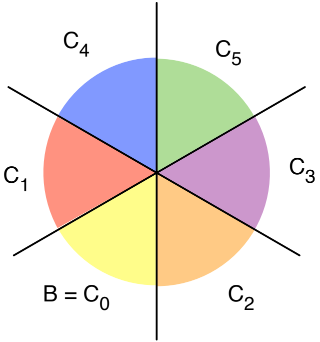
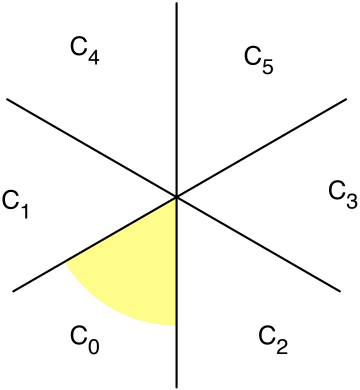


Definition: X_C



For every chamber C there a unique minimal $X_C \in \mathcal{L}(\mathscr{B})$ s.t. the set

$$\{H \in \mathscr{B} : H \supseteq X_C\}$$

separates C from the previous.

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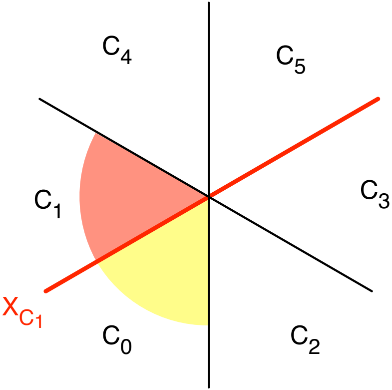

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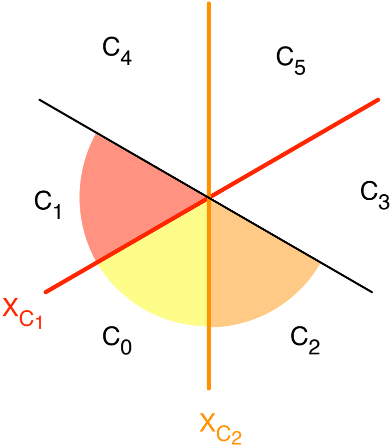

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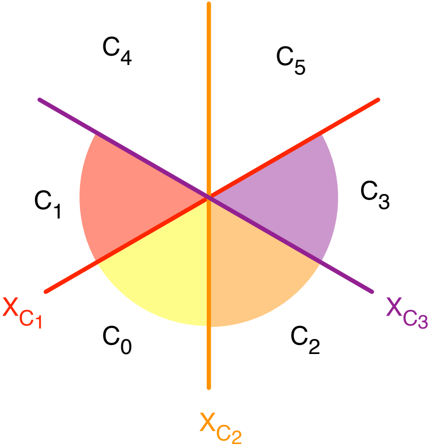

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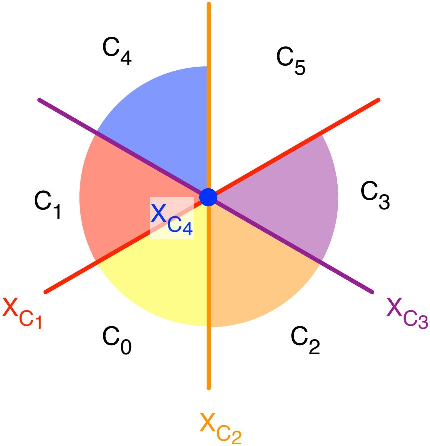

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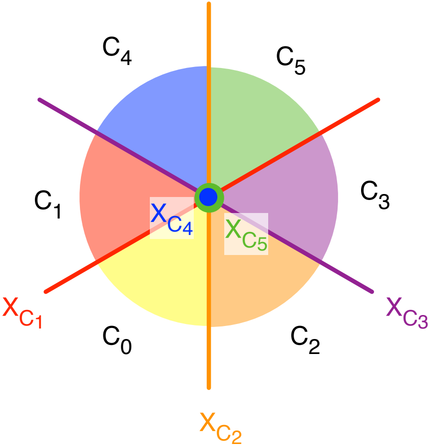

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X_C: TWO APPLICATIONS

Let $\mathscr{B}$ be a central arrangement of real hyperplanes, fix $B \in \mathcal{P}(\mathscr{B})$.

Theorem [D. '08]. *The order preserving map*

$$\phi : \mathcal{P}_B(\mathscr{B}) \to \mathcal{L}(\mathscr{B}),\ C \mapsto X_C$$

satisfies

$$|\phi^{-1}(Y)| = |\{N \in \mathrm{nbc}(\mathscr{B}) \mid \cap N = Y\}|$$

X_C: TWO APPLICATIONS

Let $\mathscr{B}$ be a central arrangement of real hyperplanes, fix $B \in \mathcal{P}(\mathscr{B})$.

Theorem [D. '08]. *The order preserving map*

$$\phi : \mathcal{P}_B(\mathscr{B}) \to \mathcal{L}(\mathscr{B}),\ C \mapsto X_C$$

satisfies

$$|\phi^{-1}(Y)| = |\{N \in \mathrm{nbc}(\mathscr{B}) \mid \cap N = Y\}|$$

Theorem [D. '08]. *There is an order preserving map*

$$\mathrm{Sal}(\mathscr{B}) \to \mathcal{P}_B(\mathscr{B})$$

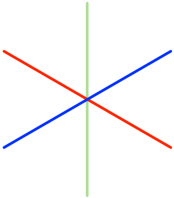
such that for the preimage $\mathcal{N}_C$ of every $C \in \mathcal{P}_B(\mathscr{B})$ we have a poset isomorphism

$$\mathcal{N}_C \simeq \mathcal{F}(\mathscr{B}^{X_C})^{op}$$

Combinatorial bookkeeping II

Fix
- $B \in \mathcal{P}(\mathscr{A}_0)$ and
- a lin. ext. of $\mathcal{P}_B(\mathscr{A}_0)$.

For all $Y \in \mathcal{C}(\mathscr{A})$, we have
- $B_Y \in \mathcal{P}(\mathscr{A}[Y])$ with $B \subseteq B_Y$
- a lin. ext. of $\mathcal{P}_{B_Y}(\mathscr{A}[Y])$.



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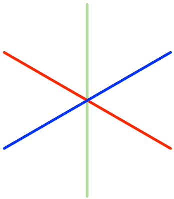

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For every $i = 0, \ldots, d$ define

$$\mathscr{Y}_i := \{(Y, C) \mid Y \in \mathcal{C}(\mathscr{A}),\ C \in \mathcal{P}_{B_Y}(\mathscr{A}[Y]),\ X_C = \max \mathcal{L}(\mathscr{A}[Y])\}.$$

Then,

$$|\mathscr{Y}_i| = |\mathscr{N}_i|.$$

COMBINATORIAL BOOKKEEPING II

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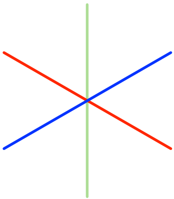

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- $B_Y \in \mathcal{P}(\mathscr{A}[Y])$ with $B \subseteq B_Y$
- a lin. ext. of $\mathcal{P}_{B_Y}(\mathscr{A}[Y])$.

Let $\mathscr{Y} := \bigcup_i \mathscr{Y}_i$. For every $(Y, C) \in \mathscr{Y}$ define a subdiagram of $\mathscr{D}$

$$\mathcal{N}_{(Y,C)} : \mathcal{F}(\mathscr{A}^Y) \to \mathbf{AC}$$
$$F \mapsto \mathcal{N}_C(\mathscr{A}[|F|]).$$

Theorem [dD '12]. *This diagram is geometric, and*

$$\operatorname{colim} \mathcal{N}_{(Y,C)} \simeq \mathcal{F}(\mathscr{A}^Y)$$

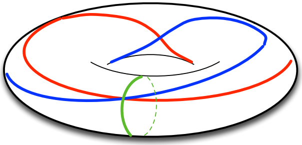
Stratification

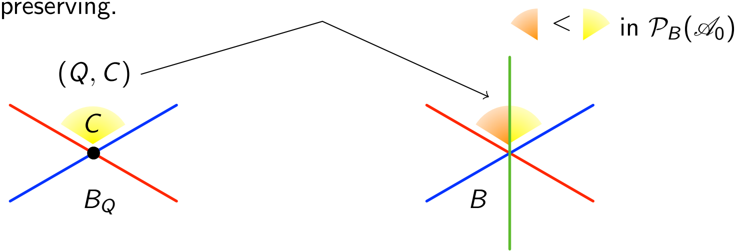
Fix
- $B \in \mathcal{P}(\mathscr{A}_0)$ and
- a lin. ext. of $\mathcal{P}_B(\mathscr{A}_0)$.

Choose total order on $\mathscr{Y}$
such that the natural map

$$\mathscr{Y} \to \mathcal{P}_B(\mathscr{A}_0)$$

is order preserving.





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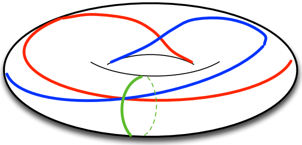

$$\mathscr{Y} \to \mathcal{P}_B(\mathscr{A}_0)$$

is order preserving.

Theorem [dD '12]. *There is a functor*

$$\Phi : \operatorname{colim} \mathscr{D} \to \mathscr{Y}$$

with

$$\Phi^{-1}(Y, C) = \operatorname{colim} \mathcal{N}_{(Y,C)}$$

Stratification

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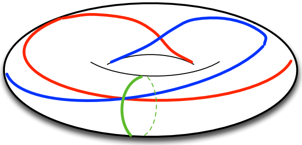

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We obtain a functor

$$\Phi : \mathrm{Sal}(\mathscr{A}) \to \mathscr{Y}$$

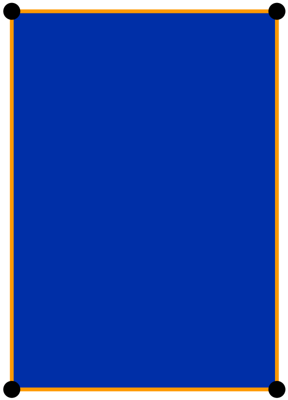
with

$$\Phi^{-1}(Y, C) = \mathcal{F}(\mathscr{A}^Y),$$

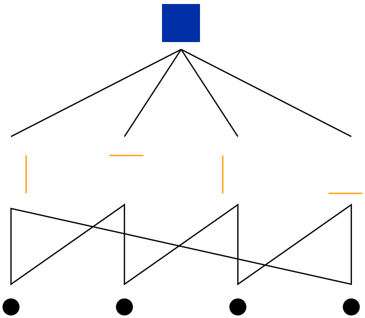
which allows us to turn to Discrete Morse Theory.

Discrete Morse theory

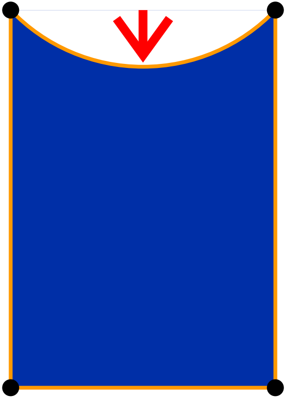
Here is a regular CW complex



with its poset of cells:

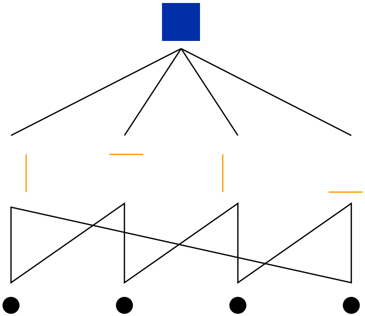


Elementary collapses...



... are homotopy equivalences.

Cells:



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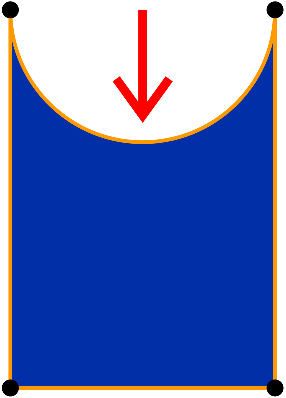

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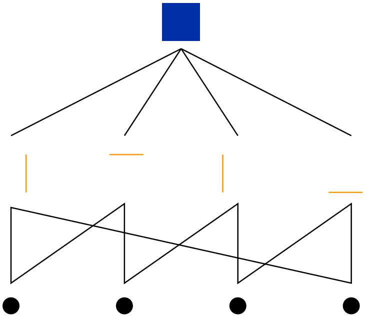

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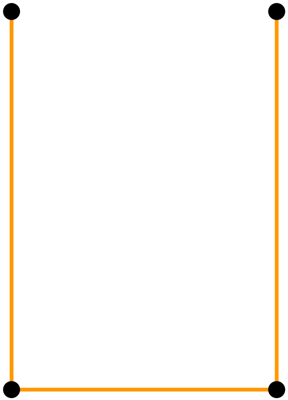

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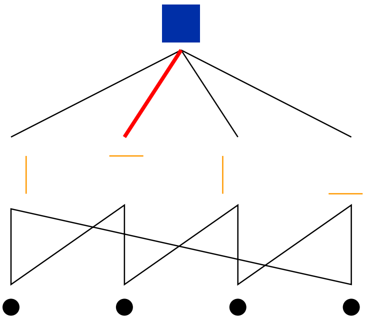

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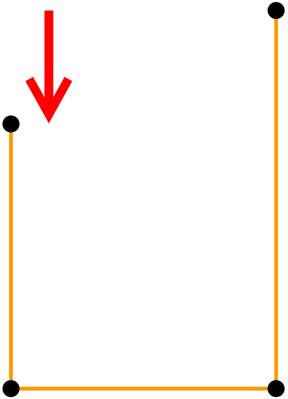

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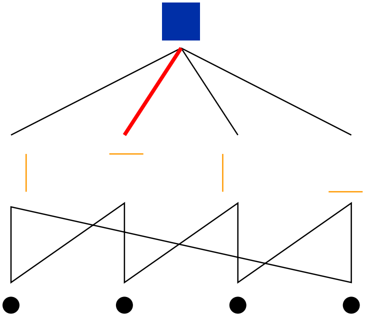

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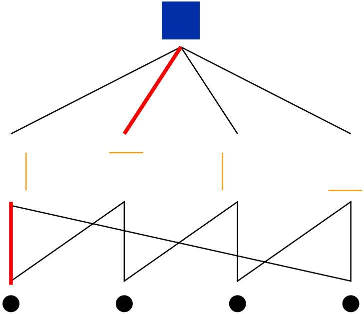

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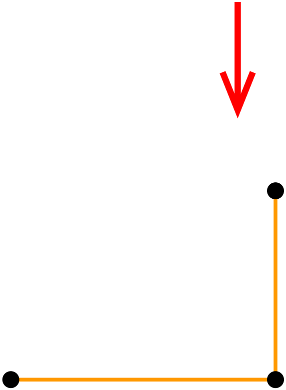

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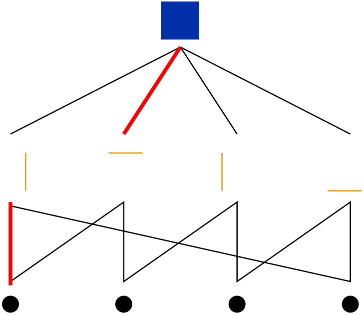

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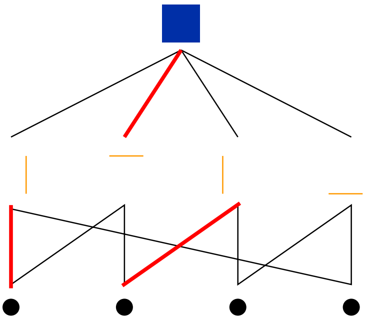

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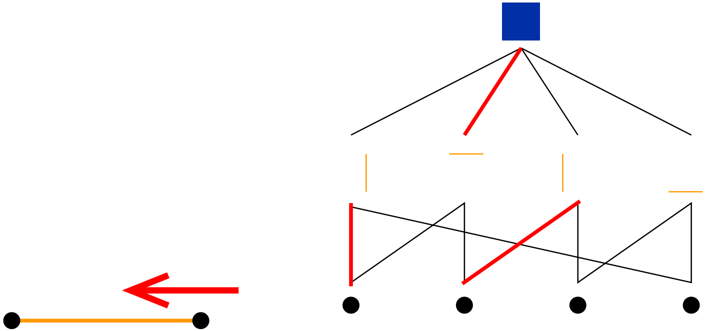

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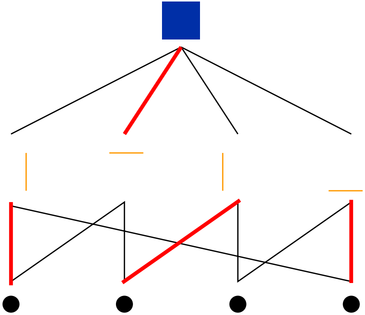

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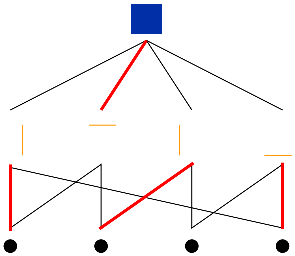
Acyclic matchings

The sequence of collapses is encoded in a matching of the Hasse diagram of the poset of cells.



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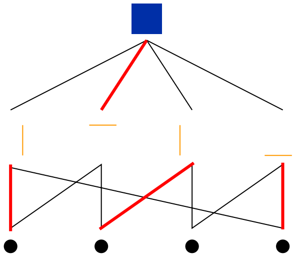

Question: Does any matchings encode such a sequence?

Acyclic matchings

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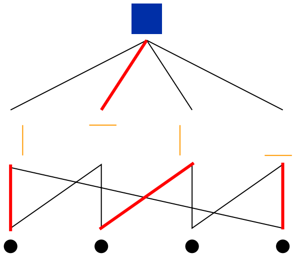
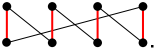


Question: Does any matchings encode such a sequence?
Answer: No. Only (and exactly) those without "cycles" like



Acyclic matchings $\leftrightarrow$ discrete Morse functions.

DMT FOR ACYCLIC CATEGORIES

Meta-Theorem [dD '12]. Discrete Morse Theory generalizes successfully to nerves of acyclic categories.

In particular, we have

- A notion of 'acyclic matching'
- A corresponding 'main theorem'
- A corresponding 'Patchwork Lemma':

DMT FOR ACYCLIC CATEGORIES

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- A notion of 'acyclic matching'
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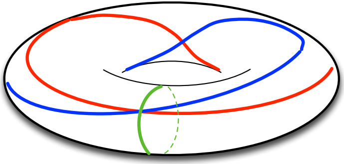
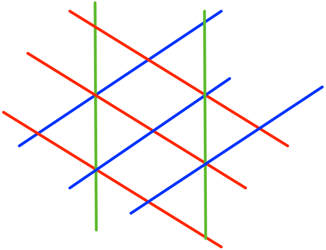
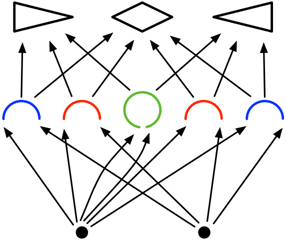
- A corresponding 'Patchwork Lemma':

  Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a functor of acyclic categories. For $b \in \mathrm{Ob}(\mathcal{B})$ let M_b be an acyclic matching of the preimage $\varphi^{-1}(b)$. Then, the union $M := \bigcup_b M_b$ is an acyclic matching of $\mathcal{A}$.

Perfect matchings for real toric arrangements







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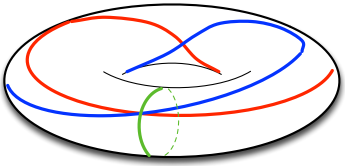

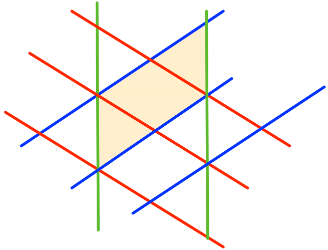

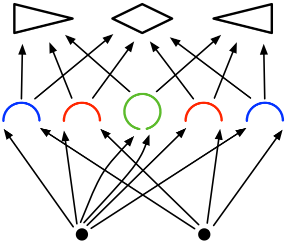

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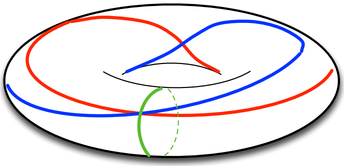

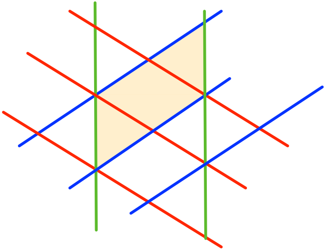

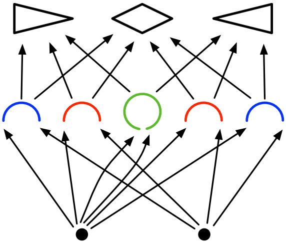

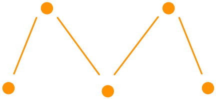

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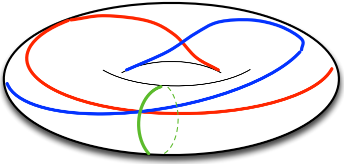

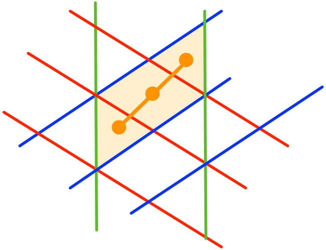

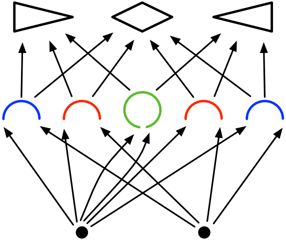

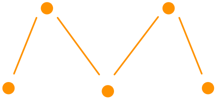

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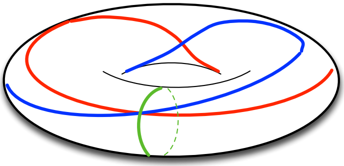

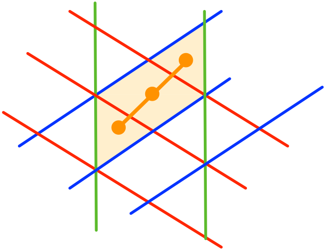

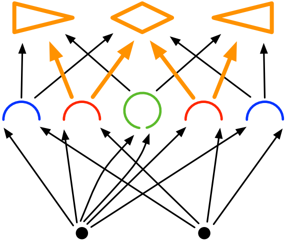

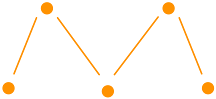

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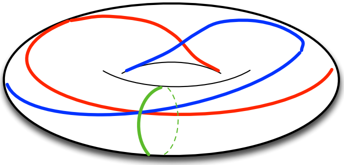

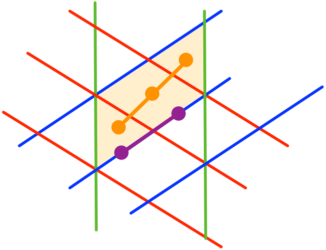

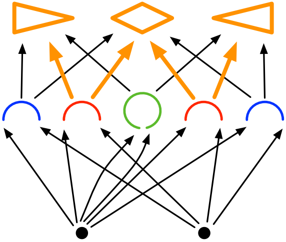

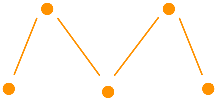

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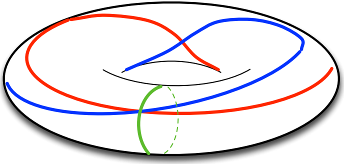

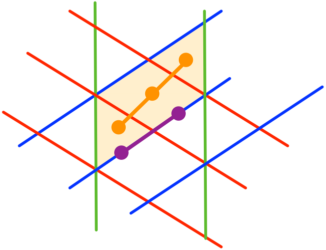

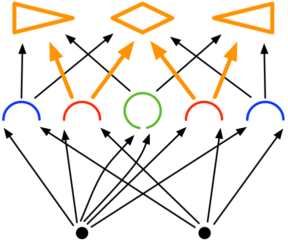

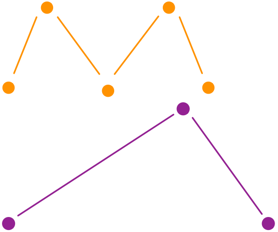

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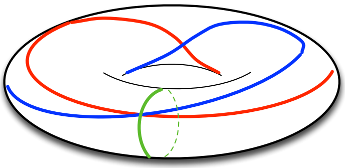

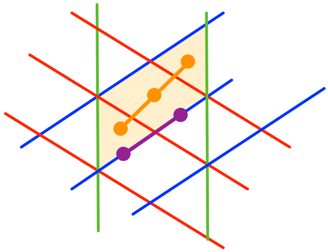

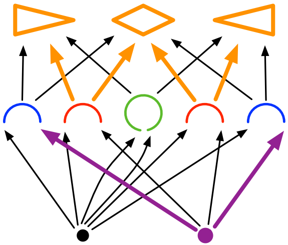

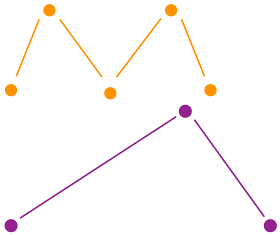

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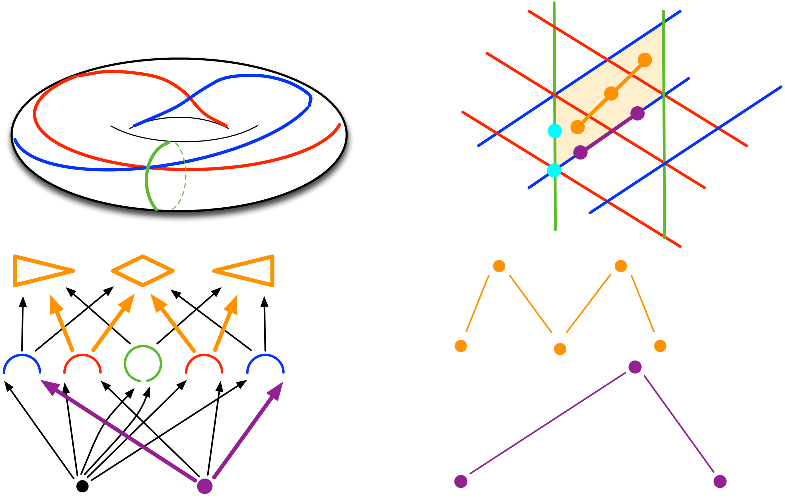
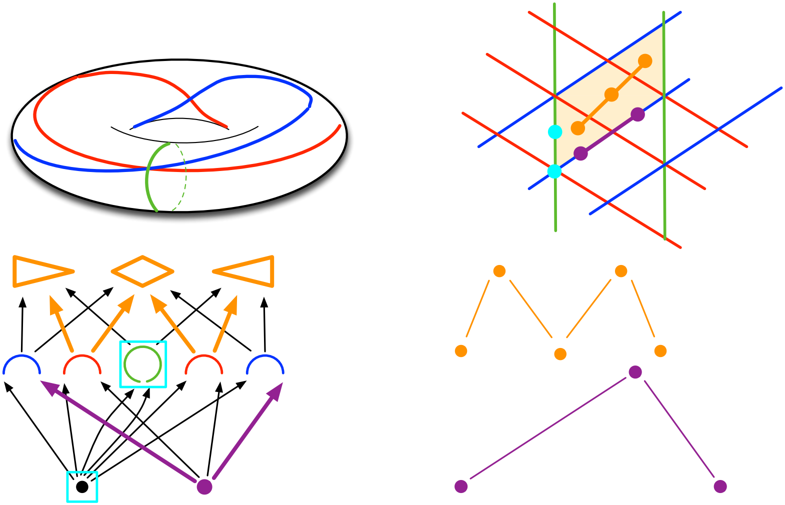


Perfect matchings for real toric arrangements



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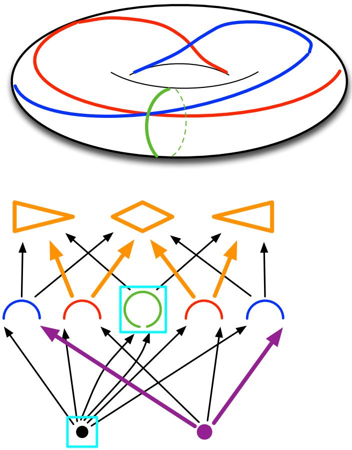

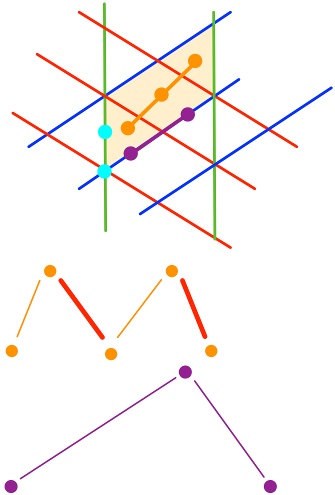

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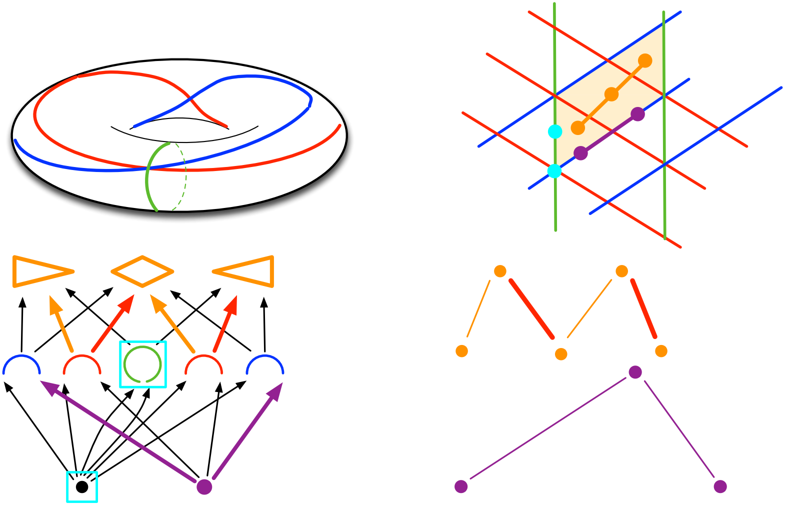

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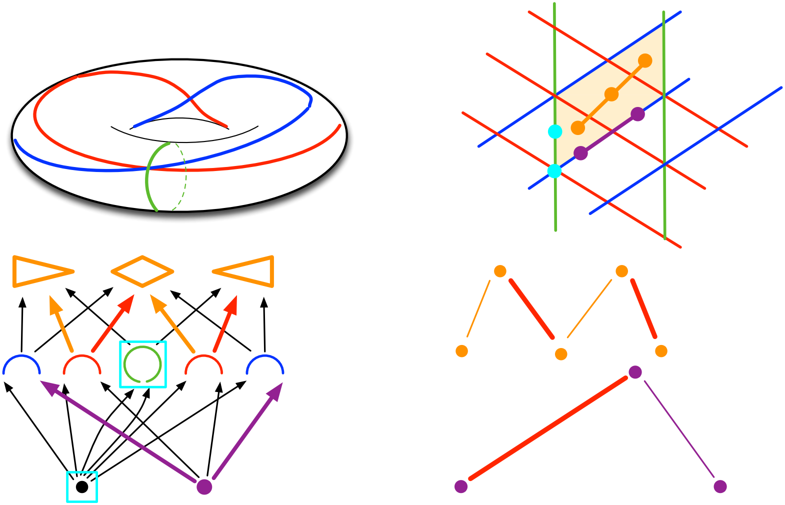

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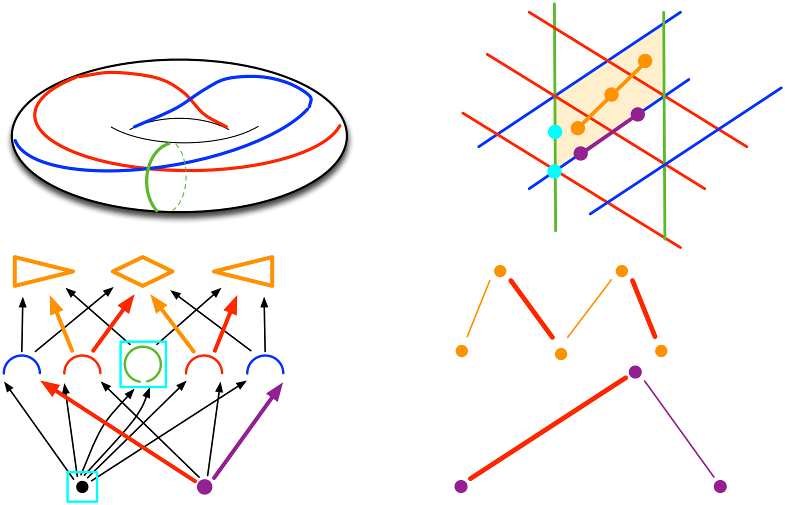

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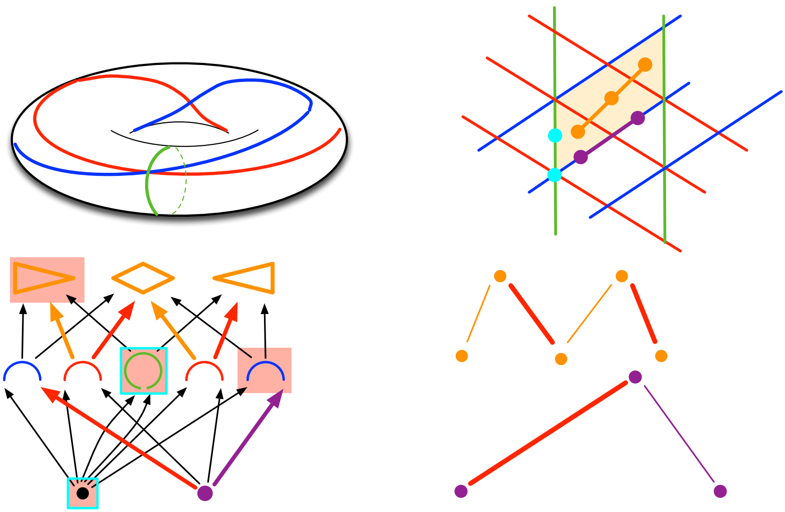

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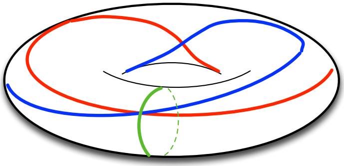

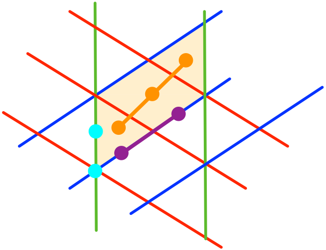

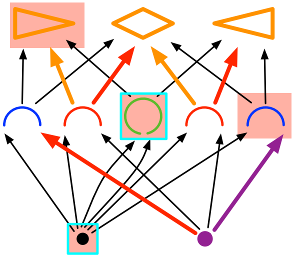

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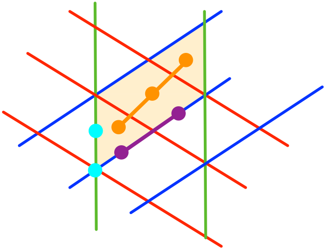
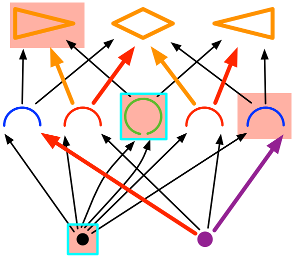


Lemma [dD '12]. *The category $\mathcal{F}(\mathscr{A})$ admits an acyclic matching with 2^d critical cells in total.*



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FIT FOR MINIMALITY

Theorem [dD '12]. *Let $\mathscr{A}$ be a complexified toric arrangement. Then $M(\mathscr{A})$ has the homotopy type of a minimal CW-complex.*

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Corollary. All cohomology modules $H^k(M(\mathscr{A}), \mathbb{Z})$ are torsion free.