Wedge operation and torus symmetries

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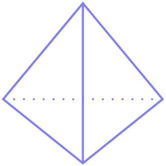
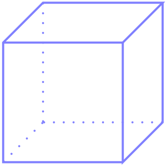
Simple polytope

A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^n$.
Let P be a convex polytope of dim n.

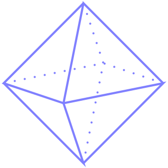
- P is simple if each vertex is the intersection of exactly n facets.
- P is simplicial if every facet is an $(n-1)$-simplex.



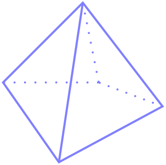
both



simple



simplicial



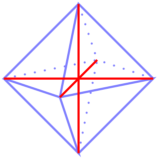
neither

Note : simple polytope $\overset{dual}{\longleftrightarrow}$ simplicial polytope

Simplicial complex

A simplicial complex K (of dim. $n-1$) on a finite set V is a collection of subsets of V satisfying

1. if $v \in V$, then $\{v\} \in K$,
2. if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$.
3. $(\max\{|\sigma| \mid \sigma \in K\} = n)$



Note : The boundary of a simplicial polytope has a simplicial complex str.

A subset $\sigma \subset V$ is called a face of K if $\sigma \in K$.
A subset $\tau \subset V$ is called a non-face of K if $\tau \notin K$.
A non-face τ is minimal if any proper subset of τ is a face of K.

Note : K is determined by its minimal non-faces.

Wedge operation

Let K : simplicial complex on $V = [m]$ and $J = (j_1, \dots, j_m) \in \mathbb{N}^m$.

Denote by $K(J)$ the simplicial complex on $j_1 + \cdots + j_m$ vertices

$$\{\underbrace{1_1, 1_2, \dots, 1_{j_1}}, \underbrace{2_1, 2_2, \dots, 2_{j_2}}, \dots, \underbrace{m_1, \dots, m_{j_m}}\}$$

with minimal non-faces

$$\{\underbrace{(i_1)_1, \dots, (i_1)_{j_{i_1}}}, \underbrace{(i_2)_1, \dots, (i_2)_{j_{i_2}}}, \dots, \underbrace{(i_k)_1, \dots, (i_k)_{j_{i_k}}}\}$$

for each minimal non-face $\{i_1, \dots, i_k\}$ of K.

The simplicial wedge operation or (simplicial) wedging of K at i is

$$\mathrm{wedge}_i(K) = K(1, \dots, 1, 2, 1, \dots, 1).$$

Note : Let I be a 1-simplex I with $\partial I = \{i_1,\ i_2\}$. Then, wedging is also defined as

$$\mathrm{wedge}_i(K) = (I \star \mathrm{Lk}_K\{i\}) \cup (\partial I \star (K \setminus \{i\})).$$

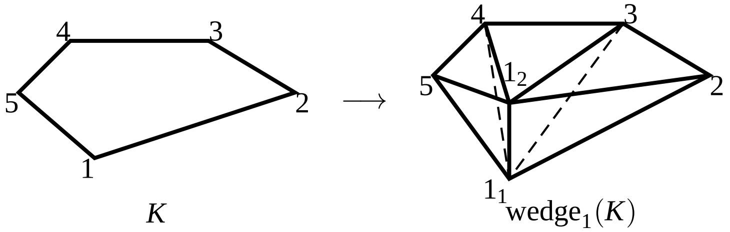
Example

K : the boundary complex of a pentagon.
Then the minimal non-faces of K are

$$\{1,3\},\{1,4\},\{2,4\},\{2,5\}, \text{ and } \{3,5\}.$$

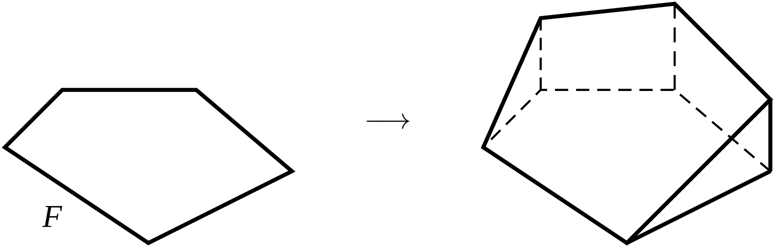
Hence, the minimal non-faces of $\mathrm{wedge}_1(K) = K(2,1,1,1,1)$ are

$$\{1_1,1_2,3\},\{1_1,1_2,4\},\{2,4\},\{2,5\}, \text{ and } \{3,5\}.$$



Example

As simple polytope...
$P \times [0, \infty) \subseteq \mathbb{R}^{n+1}$ and identify P with $P \times \{0\}$.
Pick a hyperplane H in $\mathbb{R}^{n+1}$ s.t $H \cap P = F$ and $H \cap P \times [0, \infty) \neq \emptyset$.
Then H cuts $P \times [0, \infty)$ into two parts to obtain the part $\mathrm{wedge}_F(P)$ containing P.



P $\qquad$ $\mathrm{wedge}_F(P)$

Simplicial sphere

Let K be a simplicial complex of dimension $n-1$.

1. K : simplicial sphere if its geometric realization $|K| \cong S^{n-1}$.
2. K : star-shaped if $\exists |K| \overset{emb}{\hookrightarrow} \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ s.t. any ray from p intersects $|K|$ once and only once.
3. K : polytopal if $\exists |K| \overset{emb}{\hookrightarrow} \mathbb{R}^n$ which is the boundary of a simplicial n-polytope.

simplicial complexes $\supsetneq$ simplicial spheres

$\supsetneq$ star-shaped complexes $\supsetneq$ polytopal complexes.

Proposition

1. $\mathrm{wedge}_v(K)$ *is a simplicial sphere if and only if so is* K.
2. $\mathrm{wedge}_v(K)$ *is star-shaped if and only if so is* K.
3. $\mathrm{wedge}_v(K)$ *is polytopal if and only if so is* K.

Torus manifolds

- M : connected smooth closed manifold of dim n
- $T^k = (S^1)^k$: compact abelian Lie group of rank k

T^k acts on M effectively $\Longrightarrow \dim T^k + \dim T^k_x \leq \dim M$ for any $x \in M$. Hence, if $p \in M^{T^k}$, then $\dim T^k = \dim T^k_p = k$. Hence, $2k \leq n$.

Hattori-Masuda [1] introduce one interesting class of manifolds.

Definition

A $2n$ dimensional connected smooth closed manifold M is called a torus manifold if T^n acts on M effectively with non-empty fixed point set.

[1] A. Hattori and M. Masuda, Theory of multi-fans, Osaka J. Math. 40 (2003)

Toric variety

- $\mathbb{C}^* := \mathbb{C} - O$
- a toric variety X : a normal complex algebraic variety with algebraic $(\mathbb{C}^*)^n$-action having a dense orbit

Example

$(\mathbb{C}^*)^n \curvearrowright \mathbb{C}P^n \ (= \mathbb{C}^{n+1} - O / \sim)$ defined by

$$(t_1, \dots, t_n) \cdot [z_0, z_1, \dots, z_n] = [z_0, t_1 z_1, \dots, t_n z_n].$$

- a morphism $f\colon X \to X'$: $\exists \rho\colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n'}$ s.t. $f(tx) = \rho(t) f(x)$

Fan

- $C \subset \mathbb{R}^n$ is a cone if $\exists \{v_i \in \mathbb{Z}^n\}_{1 \le i \le k}$ s.t.

$$C = \left\{ \sum_{i=1}^{k} a_i v_i \,\middle|\, a_i \ge 0 \ \forall i \right\}.$$

- A fan Δ: collection of cones in $\mathbb{R}^n$ s.t.
  1. $C \in \Delta \Longrightarrow$ faces of $C \in \Delta$
  2. $C_1,\ C_2 \in \Delta \Longrightarrow C_1 \cap C_2$ is a face of each of C_1 and C_2.
- a morphism $\Delta \to \Delta'$ is a linear map $\mathbb{R}^n \to \mathbb{R}^{n'}$ which maps $\mathbb{Z}^n \to \mathbb{Z}^{n'}$ and a cone into a cone.

Note : $\Delta = (K_\Delta, \lambda(\Delta)) := \{v_i \in \mathbb{Z}^n \mid i = 1, \ldots, m\})$.

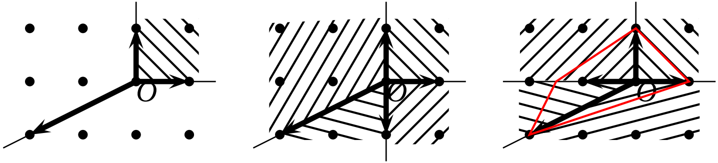
Toric manifold

Theorem (Fundamental theorem for toric varieties)

The category of toric varieties is equivalent to the category of fans.

$$X \longleftrightarrow \Delta_X$$

- X is compact $\Longleftrightarrow$ Δ_X covers R^n (complete).
- X is non-singular $\Longleftrightarrow$ every cone in Δ_X consists of unimodular rays (non-singular).
- X is projective $\Longleftrightarrow$ Δ_X is "polytopal".

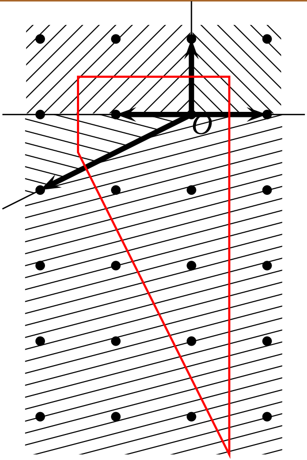


non-complete　　singular　　non-singular, polytopal

A compact non-singular toric variety is called a toric manifold.

Fan and polytope

Δ : a polytopal non-singular fan
(X_Δ : a projective toric manifold)



- $S^1 \subset \mathbb{C}^*$: a unit circle
- $T^n := (S^1)^n \subset (\mathbb{C}^*)^n$: (compact real) torus of dimension n

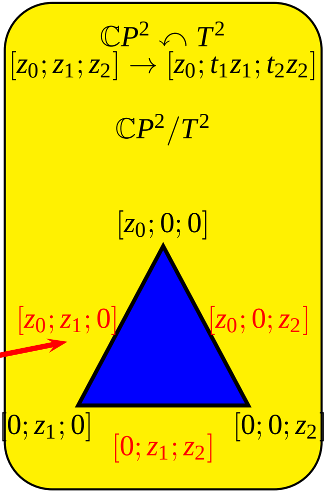
Since $(\mathbb{C}^*)^n$ acts on X_Δ, T^n also acts on X_Δ, and X_Δ/T^n is a simple polytope.

Topological analogue of toric manifolds

A quasitoric manifold M of $\dim 2n$ [2]:

- $M \curvearrowleft T^n$ locally standard, and
- $M/T^n \cong P^n$ simple polytope.

$$\begin{cases} \text{fixed point} & \longleftrightarrow \text{vertex} \\ \vdots & \qquad \vdots \\ \text{fixed by } S^1 \subset T^n & \longleftrightarrow \text{facet} \end{cases}$$



Fixed by $S^1 \times 1 \subset T^2$

[2] M. W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J. 62 (1991)

Quasitoric manifolds

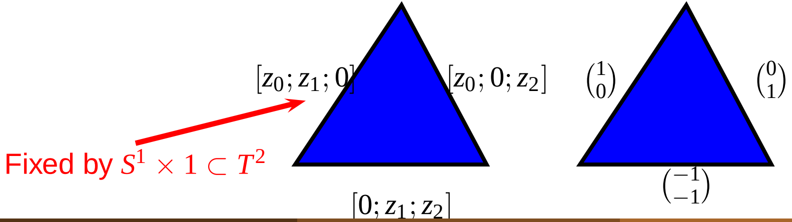
P^n : simple polytope with m facets $\mathfrak{F} = \{F_1, \dots, F_m\}$

$\lambda\colon \mathfrak{F} \to \mathrm{Hom}(S^1, T^n) \cong \mathbb{Z}^n$ characteristic function of dim n satisfying

$$F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset \Rightarrow \{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\} \text{ is basis for } \mathbb{Z}^n.$$

Theorem (Davis-Januskiewicz 1991)

All quasitoric manifolds can be indexed by (P^n, λ).



Characteristic map

Definition

A characteristic map is a pair (K, λ) of K and $\lambda \colon V(K) = [m] \to \mathbb{Z}^n$ s.t.

- $\lambda(i)$ is primitive $\forall i \in V(K)$
- $\{\lambda(i) \mid i \in \sigma\}$ is linearly independent over $\mathbb{R}$ $\forall \sigma \in K$.

Moreover,

1. (K, λ) is called complete if K is star-shaped.
2. (K, λ) is called non-singular if $\{\lambda(i) \mid i \in \sigma\}$ spans a unimodular submodule of $\mathbb{Z}^n$ of rank $|\sigma|$ $\forall \sigma \in K$.
3. (K, λ) is called positive if $\mathit{sign} \det(\lambda(i_1), \ldots, \lambda(i_n)) = o(\sigma)$ $\forall \sigma = (i_1, \ldots, i_n) \in K$, where o is the ori. of K as a simplicial mfd.
4. (K, λ) is called fan-giving if $\exists \Delta$ s.t. $(K, \lambda) = (K_\Delta, \lambda(\Delta))$.

Sometimes we call the map λ itself a characteristic map.

Topological toric manifolds

Recall that a toric manifold admits an algebraic $(\mathbb{C}^*)^n$-action.
A topological toric manifold is a closed smooth $2n$-manifold M with an effective smooth $(\mathbb{C}^*)^n$-action with some condition. [3]

Theorem (Ishida-Fukukawa-Masuda 2013)

All topological toric manifolds can be indexed by (K, λ) as T^n-manifolds, where (K, λ) is non-singular complete.

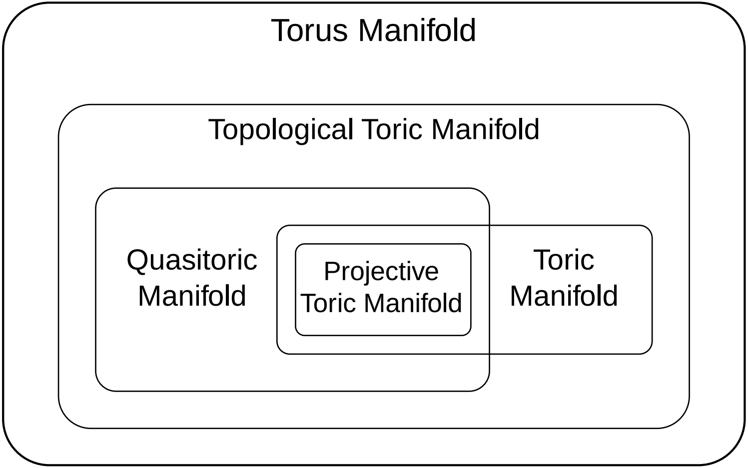
Note :

- If (K, λ) is singular, $M(K, \lambda)$ is an orbifold.
- If K is polytopal, $M(K, \lambda)$ is a quasitoric manifold.
- If K is polytopal and (K, λ) is positive, $M(K, \lambda)$ admits a T^n-equivariant almost complex structure. [4]
- If (K, λ) is fan-giving, $M(K, \lambda)$ is a toric manifold.

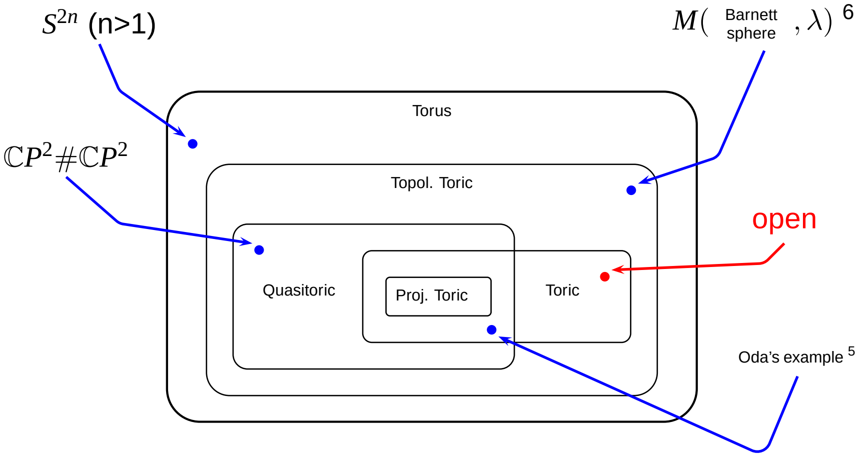
[3] H. Ishida, Y. Fukukawa, M. Masuda, Topological toric manifolds, Moscow Math. J. 13 (2013)

[4] A. A. Kustarev, Equivariant almost complex structures on quasitoric manifolds, Uspekhi Mat. Nauk 64 (2009)

Torus symmetries



Examples



[5] T. Oda, Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties, 1988.

[6] H. Ishida, Y. Fukukawa, M. Masuda, Topological toric manifolds, Moscow Math. J. 13 (2013)

Ewald and BBCG construction

Ewald found infinitely many non-projective toric manifolds. [7]

Theorem (Ewald 1986)

$$M(K,\lambda):\ \textit{toric} \quad \Longrightarrow \quad \exists\, M(\mathrm{wedge}_v(K),\lambda'):\ \textit{toric.}$$

Furthermore, $M(K,\lambda)$ *is projective* $\Leftrightarrow$ $M(\mathrm{wedge}_v(K),\lambda')$ *is projective.*

BBCG rediscovered the idea to produce infinitely many quasitoric mfds. [8]

Theorem (Bahri-Bendersky-Cohen-Gitler 2010)

$$M(K,\lambda):\ \textit{quasitoric} \quad \Longrightarrow \quad \exists\, M(\mathrm{wedge}_v(K),\lambda'):\ \textit{quasitoric.}$$

In fact, $M(K,\lambda)$ is toric $\Leftrightarrow$ $M(\mathrm{wedge}_v(K),\lambda')$ is toric.

[7] G. Ewald, Spherical complexes and nonprojective toric varieties, Discrete Comput. Geom. 1 (1986)

[8] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, Operations on polyhedral products and a new topological construction of infinite families of toric manifolds, arXiv:1011.0094 (2010)

Question

To classify toric objects, it seems natural to classify K supporting a toric object first, and find all available λ on a fixed K.

It raises the following natural question.

Question

Find all available λ on $\mathrm{wedge}_v(K)$ when K supports a toric object in each category .

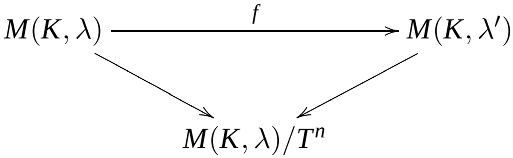
Equivalence

Let K : star-shaped simplicial sphere with $V(K) = [m]$ and $\lambda\colon V(K) \to \mathbb{Z}^n$ a characteristic function of dim n.
Denote λ by an $n \times m$ matrix

$$\lambda = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & m \\ \hline \lambda(1) & \lambda(2) & \lambda(3) & \lambda(4) & \cdots & \lambda(m) \end{array}\right)$$

Note : TFAE

- $M(K,\lambda)$ is D-J equivalent to $M(K,\lambda')$
- $\exists$ weakly equivariant $f : M(K,\lambda) \to M(K,\lambda')$ s.t.



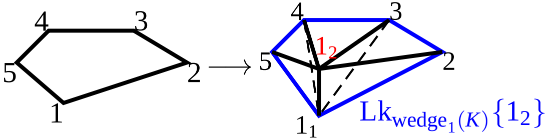
- λ' can be obtained from λ by elementary row operations.

Projected characteristic map

Let K be a simplicial complex and $\sigma \in K$.
The link of σ is

$$\mathrm{Lk}_K \sigma := \{\tau \in K \mid \sigma \cup \tau \in K,\ \sigma \cap \tau = \varnothing\}.$$

Note : $\mathrm{Lk}_{\mathrm{wedge}_v K}\{v_1\} \cong K$

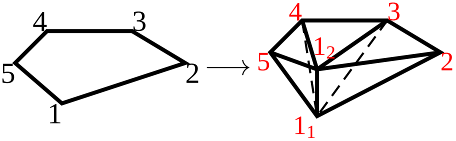


Let (K, λ) be a char. map of dim n and $\sigma \in K$.

Definition

A characteristic map $(\mathrm{Lk}_K \sigma, \mathrm{Proj}_\sigma \lambda)$, called the projected characteristic map, is defined by the map

$$(\mathrm{Proj}_\sigma \lambda)(v) = [\lambda(v)] \in \mathbb{Z}^n / \langle \lambda(w) \mid w \in \sigma \rangle \cong \mathbb{Z}^{n-|\sigma|}.$$



K $\qquad$ $\mathrm{wedge}_1(K)$

$$\lambda = \begin{pmatrix} 1_1 & 1_2 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\mathrm{Proj}_w \lambda)(v) = [\lambda(v)] \in \mathbb{Z}^n / \langle \lambda(w) \rangle \cong \mathbb{Z}^{n-1}$$

$$\mathrm{Proj}_{1_1} \lambda = \begin{pmatrix} 1_2 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix}$$

$$\mathrm{Proj}_{1_2} \lambda = \begin{pmatrix} 1_1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

$$\lambda \sim \begin{pmatrix} 1_1 & 1_2 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Main theorem

K : star-shaped simplicial sphere

Theorem

Let $(\mathrm{wedge}_v(K), \lambda)$ a characteristic map, and v_1, v_2 the two new vertices of $\mathrm{wedge}_v(K)$ created from the wedging at $v \in V(K)$.
Then λ is uniquely determined up to D-J equivalence by $\mathrm{Proj}_{v_1} \lambda$ and $\mathrm{Proj}_{v_2} \lambda$.

Theorem

1. *λ is non-singular $\Longleftrightarrow$ both $\mathrm{Proj}_{v_1} \lambda$ and $\mathrm{Proj}_{v_2} \lambda$ are non-singular.*
2. *λ is positive $\Longleftrightarrow$ both $\mathrm{Proj}_{v_1} \lambda$ and $\mathrm{Proj}_{v_2} \lambda$ are positive.*
3. *λ is fan-giving $\Longleftrightarrow$ both $\mathrm{Proj}_{v_1} \lambda$ and $\mathrm{Proj}_{v_2} \lambda$ are fan-giving.*

Main theorem

Roughly speaking..

- If one knows every topological toric manifold over K, then we know every topological toric manifold over a wedge of K.
- If one knows every quasitoric manifold over P, then we know every quasitoric manifold over a wedge of P.
- If one knows every toric manifold over K, then we know every toric manifold over a wedge of K.

Revisit Ewald and BBCG construction

For given (K, λ),

$$\lambda = \left(\begin{array}{c|ccc} v & & & \\ \hline 1 & a_2 & \cdots & a_m \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right),$$

put λ' (called the canonical extension) on $\mathrm{wedge}_v(K)$ by

$$\lambda' = \left(\begin{array}{cc|ccc} v_1 & v_2 & & & \\ \hline 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & a_2 & \cdots & a_m \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & A & \\ 0 & 0 & & & \end{array}\right).$$

Then, $\mathrm{Proj}_{v_1} \lambda = \mathrm{Proj}_{v_2} \lambda$ which are chr. maps for K. Hence, by our theorems, it implies the results by Ewald and BBCG.

Interesting class of simplicial spheres

It is natural to consider a class of (star-shaped) simplicial spheres which is closed under wedging.

Observe that

1. $K =$ boundary of $\Delta^{n_1} \times \Delta^{n_2} \times \cdots \times \Delta^{n_\ell}$

$$\Longrightarrow K = \partial\Delta^{n_1} \star \cdots \star \partial\Delta^{n_\ell}$$

$$\Longrightarrow \quad \mathrm{wedge}_v(K) = \partial\Delta^{n_1+1} \star \partial\Delta^{n_2} \star \cdots \star \partial\Delta^{n_\ell}$$

2. Define $\mathrm{Pic}\, K = |V(K)| - \dim K + 1 = m - n$. Then, $\mathrm{Pic}\, K$ is invariant under wedging.

We are interested in $\{\partial\Delta^{n_1} \star \cdots \star \partial\Delta^{n_\ell}\}$ and $\{K \mid \mathrm{Pic}\, K \leq 3\}$.

Generalized Bott manifolds

generalized Bott tower

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 \; = \{\text{a point}\},$$

where each $\pi_i \colon B_i = P(\mathbb{C} \oplus \xi_i) \to B_{i-1}$ and ξ_i is the Whitney sum of n_i complex line bundles over B_{i-1} for $i = 1, \ldots, \ell$.

We call each B_n a generalized Bott manifold.

A generalized Bott manifold is a projective toric manifold over $\partial\Delta^{n_1} \star \cdots \star \partial\Delta^{n_\ell}$.

Toric objects over $\partial\Delta^{n_1} \star \cdots \star \partial\Delta^{n_\ell}$

<u>Note</u> : one knows all toric objects over a cross-polytope (= $\partial\Delta^1 \star \cdots \star \partial\Delta^1$)[9].

Corollary

We classify all toric objects over $\partial\Delta^{n_1} \star \cdots \star \partial\Delta^{n_\ell}$. In particular, any toric manifold over $\partial\Delta^{n_1} \star \cdots \star \partial\Delta^{n_\ell}$ is a generalized Bott manifold.

The above corollary covers works by Batyrev [10], Dobrinskaya [11], and C-Masuda-Suh [12].

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[9] M. Masuda, P. E. Panov, Semifree circle actions, Bott towers and quasitoric manifolds, Sb. Math. 199 (2008)

[10] V. V. Batyrev, On the classification of smooth projective toric varieties, Tohoku Math. J. (2) 43 (1991)

[11] N. E. Dobrinskaya, Classification problem for quasitoric manifolds over a given simple polytope, Funct. Anal. and Appl. 35 (2001)

[12] S. Choi, M. Masuda, and D. Y. Suh, Quasitoric manifolds over a product of simplices, Osaka J. Math. 47 (2010)

Toric manifolds of small Picard number

We focus our interests in toric manifolds with small $\operatorname{Pic} K$.

$\operatorname{Pic} K$	K	toric manifold	projective?
1	$\partial\Delta^n$	$\mathbb{C}P^n$	Yes
2	$\partial\Delta^k \star \partial\Delta^{n-k}$	$\mathbb{C}P^k$-bundle over $\mathbb{C}P^{n-k}$	Yes
3	?	?	?
4	?	?	No

- X : toric manifold of Picard number 3
- $\Delta = (K, \lambda)$: corresponding fan ($\Rightarrow$ fan-giving non-singular)

We observe that K is obtained by a sequence of wedgings from either the octahedron $= \partial\Delta^1 \star \partial\Delta^1 \star \partial\Delta^1$ or the pentagon P_5. [13]

[13] J. Gretenkort, P. Kleinschmidt, B. Sturmfels, *On the existence of certain smooth toric varieties*, Biscrete Comput. Geom. 5 (1990)

Toric manifolds of small Picard number 3

- from $\partial\Delta^1 \star \partial\Delta^1 \star \partial\Delta^1$: X is a generalized Bott manifold.
- from P_5 :

Up to rotational symmetry of P_5 and basis change of $\mathbb{Z}^2$, any fan-giving non-singular chr. map is described by

$$\lambda_d := \begin{pmatrix} 1 & 0 & -1 & -1 & d \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix}$$

for an arbitrary $d \in \mathbb{Z}$.

Corollary

We classify all toric manifolds of Picard number 3.

Note : It reproves the result of Batyrev[14].

[14] V. V. Batyrev, On the classification of smooth projective toric varieties, Tohoku Math. J. (2) 43 (1991)

Example

P_5 : pentagon on $\{1,2,3,4,5\}$
Consider $\mathrm{wedge}_3\, P_5$ and assume that $\mathrm{Proj}_{3_1}(\lambda) = \lambda_d$.

$$\lambda = \begin{pmatrix} \color{red}{1} & \color{red}{2} & \color{red}{3_1} & \color{red}{3_2} & \color{red}{4} & \color{red}{5} \\ \hline 1 & 0 & 0 & -1 & -1 & d \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & n_3 & n_4 & n_5 \end{pmatrix}$$

Now consider $\mathrm{Proj}_{3_2}(\lambda)$.

$$\lambda \sim \begin{pmatrix} \color{red}{1} & \color{red}{2} & \color{red}{3_1} & \color{red}{3_2} & \color{red}{4} & \color{red}{5} \\ \hline -n_3 & 0 & -1 & 0 & n_3 - n_4 & -n_5 - dn_3 \\ n+3+1 & 1 & 1 & 0 & n_4 - n_3 - 1 & -1 + n_5 + dn_3 + d \\ n_3 + 1 & 0 & 1 & -1 & n_4 - n_3 - 1 & n_5 + dn_3 + d \end{pmatrix}$$

Hence,

$$\mathrm{Proj}_{3_2}(\lambda) = \left(\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3_1} & \color{red}{4} & \color{red}{5} \\ \hline -n_3 & 0 & -1 & n_3 - n_4 & -n_5 - dn_3 \\ n_3 + 1 & 1 & 1 & n_4 - n_3 - 1 & -1 + n_5 + dn_3 + d \end{array}\right)$$

$\therefore n_3 = -1, n_5 = 0, (1-d)n_4 = 0$.

Hence, all toric manifolds over $\mathrm{wedge}_3 P_5$ whose $\mathrm{Proj}_{3_1} \lambda = \lambda_d$ are of the form, for $d, e \in \mathbb{Z}$,

$$\left(\begin{array}{cccccc} \color{red}{1} & \color{red}{2} & \color{red}{3_1} & \color{red}{3_2} & \color{red}{4} & \color{red}{5} \\ \hline 1 & 0 & 0 & -1 & -1 & d \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cccccc} \color{red}{1} & \color{red}{2} & \color{red}{3_1} & \color{red}{3_2} & \color{red}{4} & \color{red}{5} \\ \hline 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & e & 0 \end{array}\right).$$

By consideration of rotation, one can find all toric manifolds over $\mathrm{wedge}_3 P_5$.

Classification

$m-n$	complex	toric manifold	projective?
1	n-simplex $\partial\Delta^n$	$\mathbb{C}P^n$	Yes
2	$\partial\Delta^k \star \partial\Delta^{n-k}$	gen. Bott manifold	Yes
3	$\partial\Delta^{n_1} \star \partial\Delta^{n_2} \star \partial\Delta^{n_3}$	gen. Bott manifold	Yes
	$P_5(a_1,\dots,a_5)$	Done	?
4	?	?	No

Table : Classification complete for Picard number 3.

Projectivity

Unfortunately, our theorem does not hold in the category of projective toric manifolds in general.

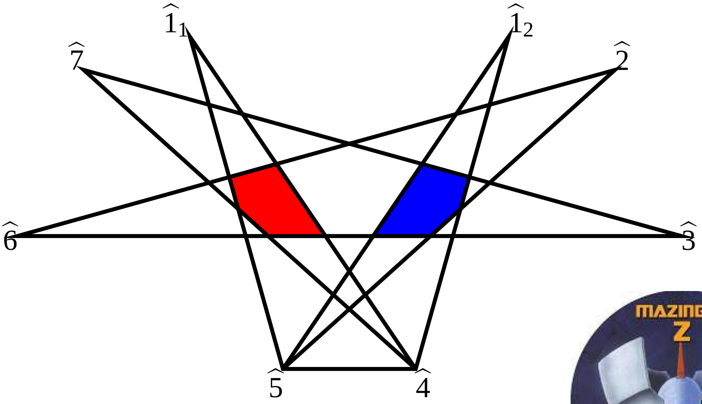
Example

Let $P_{[7]}$ be the cyclic 4-polytope with 7 vertices. Define $(\mathrm{wedge}_1 P_{[7]},\ \lambda)$ by the matrix

$$\lambda = \begin{pmatrix} 1_1 & 1_2 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline -16 & 16 & -1 & 0 & 0 & 0 & 0 & 1 \\ -33 & 83 & -6 & 0 & 0 & 0 & 1 & 0 \\ -37 & 127 & -10 & 0 & 0 & 1 & 0 & 0 \\ -33 & 123 & -10 & 0 & 1 & 0 & 0 & 0 \\ -13 & 63 & -6 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is fan-giving and singular. Then $M(\mathrm{wedge}_1 P_{[7]},\ \lambda)$ is not projective although its projections are projective.

Figure : A Shephard diagram for non-projective one over $\mathrm{wedge}_1 P_{[7]}$.



Projectivity

However, our theorem holds in the projective category with some assumptions.

Theorem

Assume $\operatorname{Pic} K = 3$. *Let* $(\mathrm{wedge}_v(K), \lambda)$ *be a fan-giving non-singular characteristic map. Then*

$$M(\mathrm{wedge}_v(K), \lambda) \text{ is projective} \iff \text{both } M(K, \mathrm{Proj}_{v_1} \lambda) \text{ and } M(K, \mathrm{Proj}_{v_2} \lambda) \text{ are projective.}$$

Note : all toric manifolds over a pentagon are projective.

Corollary

All toric manifolds of $\operatorname{Pic} 3$ *are projective.*

The above result greatly improves the proof by Kleinschmidt-Stumfels[15].

[15] P. Kleinschmidt, B. Sturmfels, Smooth toric varieties with small picard number are projective, Topology 30 (1991)

Real analogues

- M : a toric variety of complex dimension n.
  Then, $\exists$ a canonical involution ι on M, and M^ι form a real subvariety of real dimension n, called a real toric variety
- Similarly, "real" versions of topol. toric and quasitoric manifolds are real topological toric manifolds and small covers, resp.

Such real analogues of toric objects can be described as a $\mathbb{Z}_2$-version of (K, λ), that is, $\lambda : V(K) \to \mathbb{Z}_2^n$.

Theorem

λ is uniquely determined by $\operatorname{Proj}_{v_1} \lambda$ and $\operatorname{Proj}_{v_2} \lambda$. Furthermore, λ is non-singular if and only if so are $\operatorname{Proj}_{v_1} \lambda$ and $\operatorname{Proj}_{v_2} \lambda$.

Corollaries

We have the following corollaries.

1. We have all classification of toric objects over K with $\operatorname{Pic} K = 3$.
2. The number of real toric over $P_5(a_1, a_2, a_3, a_4, a_5)$ is

$$2^{a_1+a_4-1} + 2^{a_2+a_5-1} + 2^{a_3+a_1-1} + 2^{a_4+a_2-1} + 2^{a_5+a_3-1} - 5.$$

3. The number of small cover over $P_{[7]}(J)$ is 2 for any J.
4. The lifting problem holds for $\operatorname{Pic} K \leq 3$. That is, any small cover is realized as fixed points of the conjugation of a quasitoric manifold if $\operatorname{Pic} K \leq 3$. Indeed, all small covers over $P_5(J)$ are real toric.
5. By applying the Suciu-Trevisan formula, we compute rational Betti numbers of toric objects over $P_5(J)$.

Further questions

1. Does our theorem hold in the projective non-singular category?
2. Classify and study (real) toric objects over K with $\operatorname{Pic} K = 4$.
(The only thing what we have to do is to characterize all (star-shaped, polytopal) seed simplicial spheres of Picard number 4. It means that if K supports toric objects, then $K = S(J)$ for some seed S and $J \in \mathbb{N}^m$.)

Question (Cohomological rigidity problem)

M, N : *toric manifolds*

$$M \overset{diff}{\cong} N \quad \overset{?}{\Longleftrightarrow} \quad H^*(M) \cong H^*(N)$$

3. Classify toric objects over K with $\operatorname{Pic} K = 3$ topologically.
Note : The answer to the problem for $\operatorname{Pic} K \leq 2$ is affirmative.[16]
4. $\ldots$

[16] S. Choi, M. Masuda, D. Y. Suh, Topological classification of generalized Bott towers, Trans. Amer. Math. Soc. 362 (2010)

The end of the talk

Thank you!

Mulţumesc / 감사합니다