Braid cohomology, principal congruence subgroups and geometric representations

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joint work with
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Joint AMS-RMS Meeting
Alba Iulia, June 29th 2013

The braid group

Definition

The n-th braid group B_n is the fundamental group of the space of unordered n-tuples of distinct points in $\mathbb{C}$.

$$B_n := \pi_1 \left(\mathbb{C}^n \setminus \bigcup_{i<j} \{z_i = z_j\} \Big/ \mathfrak{S}_n \right)$$

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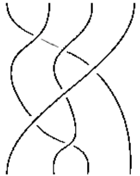

This is a braid on 4 strands.

The standard presentation of the braid group

The braid group on $n+1$ strands has a presentation given by generators and relations:

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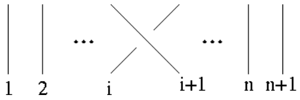
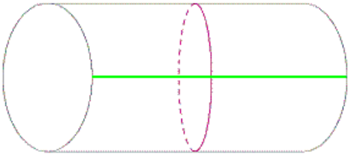
The generator σ_i corresponds to the twist



Dehn twist

Let S be an oriented surface and a a simple closed curve in S. We call D_a the Dehn twist along a.



If two simple curves a, b do not intersect, the corresponding Dehn twists commute $D_a D_b = D_b D_a$. When they intersect in one point, the associated Dehn twists satisfy the braid relation $D_a D_b D_a = D_b D_a D_b$.

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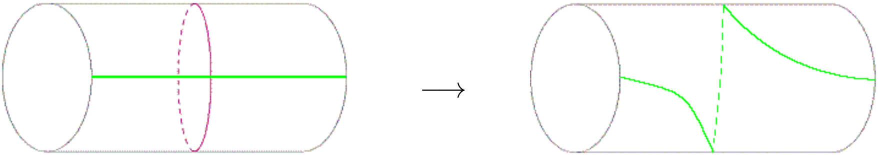

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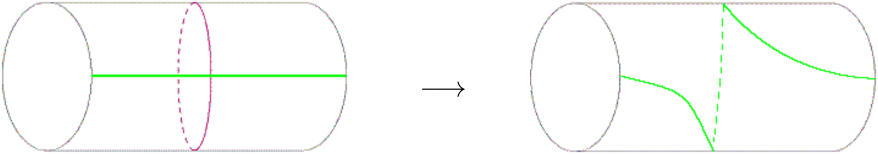

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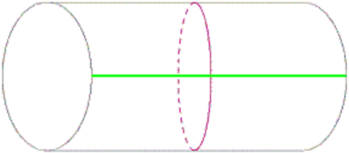

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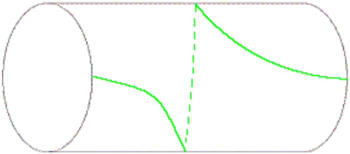

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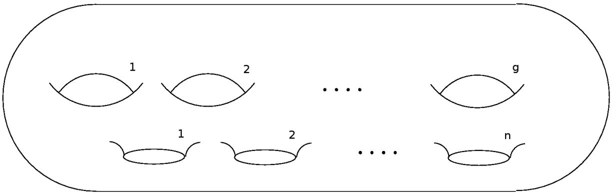
The mapping class group

Let $S_{g,n}$ be an oriented surface of genus g, with n boundary components.



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We call $MCG(S_{g,n})$ the mapping class group of $S_{g,n}$, that is the group of isotopy classes of orientation preserving diffeomorphisms of $S_{g,n}$ that fix the boundary pointwise.

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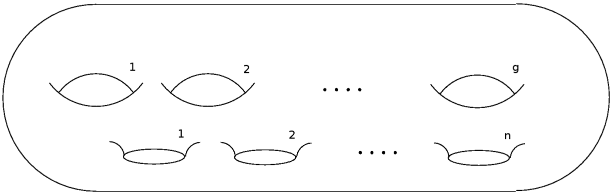

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Geometric representation

We can define standard geometric embeddings $\phi : B_{2g+1} \to MCG(S_{g,1})$ and $\phi : B_{2g+2} \to MCG(S_{g,2})$ mapping the standard braid generators to Dehn twist

and hence there is an action on the H_1 of the surface that preserves the intersection form.

$$B_{2g+1} \to Aut(H_1(S_{g,1};\mathbb{Z}), < \cdot, \cdot >) = Sp_{2g}(\mathbb{Z})$$

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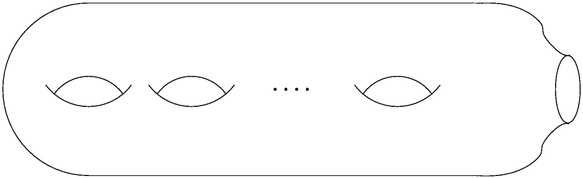

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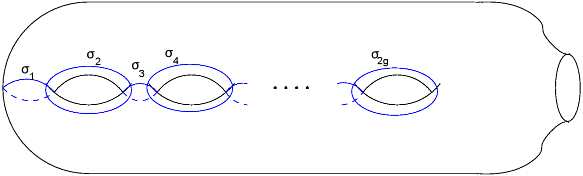

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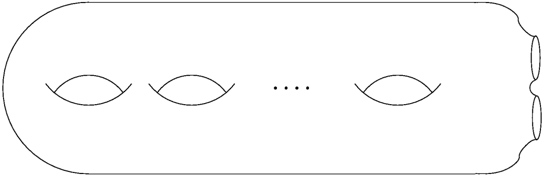

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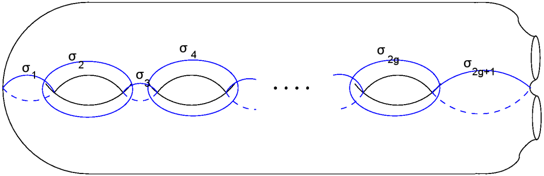

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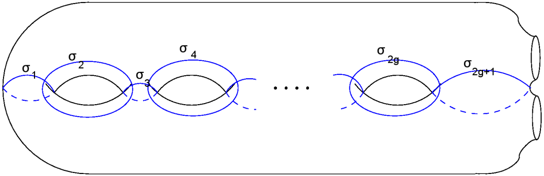

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Polynomial extension

The previous action naturally extends to the symmetric algebra with $\mathbb{Z}$-linear automorphisms that preserve the degree.

$$H_1(S_{g,1}, \mathbb{Z})^* =< x_1, y_1, \ldots, x_g, y_g >$$

$$M = \mathbb{Z}[x_1, y_1, \ldots, x_g, y_g]$$

$$B_{2g+1} \to Aut_{\mathbb{Z}}(\mathbb{Z}[x_1, y_1, \ldots, x_g, y_g])$$

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Another point of view (in a special case)

Let T^2 be the 2-dimensional compact torus.

Definition

$Diff_+(T^2)$ is the group of orientation preserving diffeomorphisms of the torus and $Diff_0 T^2$ is the connected component of the identity.

We have the exact sequence

$$1 \to Diff_0(T^2) \to Diff_+(T^2) \to SL_2(\mathbb{Z}) \to 1$$

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Homotopy equivalences

Theorem

The inclusion $T^2 \hookrightarrow Diff_0(T^2)$ is an homotopy equivalence.

As a consequence we have the homotopy equivalences

$$BDiff_0(T^2) \simeq BT^2 \simeq (\mathbb{CP}^\infty)^2$$

and the cohomology of this space is

$$M := H^*(BDiff_0(T^2); \mathbb{Z}) = \mathbb{Z}[x, y]$$

where x, y are generators in degree 2. We call M^q the homogeneous component of M of degree q.

The group $SL_2(\mathbb{Z})$ acts on $\mathbb{Z}[x, y]$ extending the standard action on $< x, y >$.

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The spectral sequence

From the fibration $BDiff_0(T^2) \hookrightarrow BDiff_+(T^2) \rightarrow BSL_2(\mathbb{Z})$ we get the Serre spectral sequence

$$E_2^{i,j} = H^i(SL_2(\mathbb{Z}); M^j) \Rightarrow H^{i+j}(BDiff_+(T^2); \mathbb{Z})$$

Theorem

The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product

$$\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6.$$

Corollary

The spectral sequence above collapses if we tensor the coefficients by a ring R such that 2 and 3 are invertible in R.

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A central extension

There is an extension

$$1 \to \mathbb{Z} \to B_3 \xrightarrow{\psi} SL_2(\mathbb{Z}) \to 1.$$

defined by $\psi : \sigma_1 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \psi : \sigma_2 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$

The kernel of ψ is the index 2 subgroup of the center of B_3. The map ψ induces an action of B_3 on $(\mathbb{CP}^\infty)^2$. Define the Borel constructions $X := EB_3 \times_{B_3} (\mathbb{CP}^\infty)^2$ that fits into the fibration

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Results

In '88 Furusawa, Tezuka, Yagita computed the cohomology of $SL_2(\mathbb{Z})$ with coefficients in the module $\mathbb{Q}[x, y]$ and $\mathbb{Z}_p[x, y]$ for any prime p.

We compute the cohomology of $SL_2(\mathbb{Z})$ and B_3 with coefficients in the module $M = \mathbb{Z}[x, y]$.

Results

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We compute the cohomology of $SL_2(\mathbb{Z})$ and B_3 with coefficients in the module $M = \mathbb{Z}[x,y]$.

Principal congruence subgroups

Definition

The *principal congruence subgroup of level* n, $\Gamma(n) \subset SL_2(\mathbb{Z})$ is the kernel of the mod-n reduction map

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_n)$$

and the group $B_{\Gamma(n)}$ is the subgroup of B_3 that is the counter-image of $\Gamma(n)$ with respect to the projection $\psi : B_3 \to SL_2(\mathbb{Z})$.

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Principal congruence subgroups - II

The group $SL_2(\mathbb{Z}_2)$ is the symmetric group Σ_3 on three elements. The group $B_{\Gamma(2)} \subset B_3$ is the kernel of the map $B_3 \to \Sigma_3$ and hence is the pure braid group P_3 on three strands.

By the Kurosh subgroup Theorem, $\Gamma(2) = F_2 \times \mathbb{Z}_2$.
For $n > 2$ the group $\Gamma(n)$ is a free, finitely generated.

We compute the cohomology with coefficients in the module $M = \mathbb{Z}[x, y]$ also for the subgroups $\Gamma(n)$ and $B_{\Gamma(n)}$.

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Modular forms

Let k be a positive integer and f an holomorphic form on the upper half-plane $\mathbb{H} \cup \{\infty\}$.

Definition

The function f is an *cusp integral modular form of weight* k (w.r. to $SL_2(\mathbb{Z})$) if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \forall k \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

and $f(\infty) = 0$.

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We call $\mathcal{M}_k^0$ the space of cusp modular forms of weight k.

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Eicher-Shimura isomorphism

Theorem

For k odd the group $H^i(SL_2(\mathbb{Z}); M^{2k} \otimes \mathbb{R})$ is always trivial.
For k even we have:

$$H^i(SL_2(\mathbb{Z}); M^{2k} \otimes \mathbb{R}) = \begin{cases} \mathcal{M}^0_{k+2} \oplus \mathbb{R} & \text{if } i = 1 \text{ and } k \geq 1 \\ 0 & \text{if } i > 0 \text{ or } i = 0 \text{ and } k > 0 \\ \mathbb{R} & \text{if } i = k = 0. \end{cases}$$

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Recall that M is trivial in odd dimension.

Divided polynomial algebra

Definition (Divided polynomial algebra)

Let $\Delta[x]$ be the sub-$\mathbb{Z}$-module of $\mathbb{Q}[x]$ generated by the elements $x_n := \frac{x^n}{n!}$, for $n \in \mathbb{N}$. For any ring R we define $\Delta_R[x] := \Delta[x] \otimes_{\mathbb{Z}} R$.

The module $\Delta[x]$ is closed by multiplication and the product satisfy the relation $x_i x_j = \binom{i+j}{i} x_{i+j}$. We define in $\Delta[x]$ the ideal $I_p := (p^{v_p(n)+1} x_n,\ \text{for } n \in \mathbb{N})$ where v_p is the p-adic additive valuation.

Definition (p-local divided polynomial algebra)

$$\Delta_p[x] := \Delta[x]/I_p.$$

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Divided polynomials and torsion

The element x_{p^n} generate in $\Delta_p[x]$ a submodule isomorphic to $\mathbb{Z}_{p^{n+1}}$.

Definition

We define also $\Delta^+[x]$ as the submodule of elements with zero constant term.
In more variables we define $\Delta_p[x, y] := \Delta_p[x] \otimes \Delta_p[y]$.

Theorem

$$\Delta_p^+[x] = \Delta^+_{\mathbb{Z}_{(p)}}[x]/(px).$$

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Cohomology of $SL_2(\mathbb{Z})$

Theorem (-, Cohen, Salvetti)

$H^1(SL_2(\mathbb{Z}); M)_{(p)} = \Delta_p^+[\mathcal{P}_p, \mathcal{Q}_p]$
where $\deg \mathcal{P}_p = 2(p+1)$ *and* $\deg \mathcal{Q}_p = 2p(p-1)$.

Theorem (-, C, S)

For $i > 1$ *the cohomology* $H^i(SL_2(\mathbb{Z}); M^q)$ *is* 2*-periodic in* i*. The free part is trivial and only* $2, 4$ *and* 3 *torsion appear.*
$H^{2i}(SL_2(\mathbb{Z}); M^{8q})$ *contains one submodule isomorphic to* $\mathbb{Z}_4$*. All the others groups are direct sum of modules isomorphic to* $\mathbb{Z}_2$ *and* $\mathbb{Z}_3$*.*
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Cohomology of B_3

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$H^1(B_3; M)_{(p)} = H^1(SL_2(\mathbb{Z}); M)_{(p)}$;
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$H^2(B_3; M)_{(p)} = H^1(SL_2(\mathbb{Z}); M)_{(p)}$ for any prime $p \geq 5$.

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$H^2(B_3; M)_{(2)} = \Delta_2^+[\mathcal{P}_2, \mathcal{Q}_2] \oplus \mathbb{Z}_2[\overline{\mathcal{Q}}_2]/\sim$
with $\frac{\mathcal{Q}_2^n}{n!} \sim 2\overline{\mathcal{Q}}_2^n$;
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Cohomology of $\Gamma(2)$

The $\Gamma(n)$-invariants in M can be easily computer with a generalization of Dickson invariant theory.

Theorem (-, C, S)

Let F_2 be the subgroup of $SL_2(\mathbb{Z})$ freely generated by s_1^2, s_2^2. The following isomorphisms hold.

For even n $H^1(\Gamma(2); M_n) = H^1(F_2; M_n)$,
and for $i > 0$ $H^{2i}(\Gamma(2); M_n) = H^0(F_2; M_n \otimes \mathbb{Z}_2) = M_n \otimes \mathbb{Z}_2$,
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Cohomology of $\Gamma(n)$

Schreier index formula allows to compute the rank of $H^*(\Gamma(m); M_n \otimes \mathbb{Q})$. [Details]

Theorem (-, C, S)

Let p be a prime number and $m > 1$ an integer.
If $p \nmid m$
the p-torsion component of $H^1(\Gamma(m); M_n)$ is given by:
$H^1(\Gamma(m); M_n)_{(p)} = H^1(SL_2(\mathbb{Z}); M_n)_{(p)} = \Delta_p^+[\mathcal{P}_p, \mathcal{Q}_p]_{\deg = n}$.
If $p \mid m$, suppose $p^a \mid m, p^{a+1} \nmid m$. Then we have
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Cohomology of $B_{\Gamma(n)}$

Theorem (-, C, S)

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Let $n > 0$; *for even* n; $H^0(B_{\Gamma(2)};M_n) = H^0(\Gamma(2);M_n)$,

$H^1(B_{\Gamma(2)};M_n) = H^2(B_{\Gamma(2)};M_n) = H^1(\Gamma(2);M_n)$

for odd n; $H^0(B_{\Gamma(2)};M_n) = 0$,

$H^1(B_{\Gamma(2)};M_n) = H^1(\Gamma(2);M_n) = M_n \otimes \mathbb{Z}_2$,

$H^2(B_{\Gamma(2)};M_n) = H^2(\Gamma(2);M_n) = (M_n \oplus M_n) \otimes \mathbb{Z}_2$

for any $m > 2$, *for any* n :

$H^*(B_{\Gamma(m)};M_n) = H^*(\Gamma(m);M_n) \otimes H^*(\mathbb{Z};\mathbb{Z})$.

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for any $m > 2$, for any n :

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Cohomology of $B_{\Gamma(n)}$

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Methods

- $SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$;
- Dickson's invariant theory for $SL_2(\mathbb{Z})$;
- explicit computations for $H^*(G; M)$, $G = \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6$;
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Cohen, Moore and Neisendorfer conjectured the existence of a p-local fibration

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Theorem

The reduced cohomology of the space $T_p(2n+1)$ is given by:

$$\overline{H}^i(T_p(2n+1); \mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}/p^r & \text{if } i = 2np^{r-1}k, p \nmid k; \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem (-,C, S)

Let $p \geq 5$ be a prime.

a) $H^*(EB_3 \times_{B_3} (\mathbb{CP}^\infty)^2; \mathbb{Z})_{(p)} = H^*(S^1 \times BDiff_+(T^2); \mathbb{Z})_{(p)}$

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Question

Is there any topological explanation for the isomorphism above?

Thank you for your attention!

Theorem

The group $H^0(\Gamma(m); M_n)$ is isomorphic to M_0 for $n = 0$ and is trivial for $n > 0$.

Theorem

Let $m > 2$ be an integer that factors as $m = p_1^{a_1} \cdots p_k^{a_k}$. The cardinality of $SL_2(\mathbb{Z}_m)$ is given by $d = \prod_i p_i^{(a_i-1)3} p_i(p_i^2 - 1)$ and if we define $i = \frac{d}{p_1(p_1^2-1)}$ then $\Gamma(m)$ is a free group of rank

$$r = \begin{cases} i/2 + 1 & \text{if } p_1 = 2 \\ i(p_1(p_1^2 - 1) - 1) + 1 & \text{if } p_1 > 2 \end{cases}$$

The rank of the group $H^1(\Gamma(m); M_n \otimes \mathbb{Q})$ is r, for $n = 0$ and $(r-1)(n+1)$ for $n > 0$.

[Back]