Braid cohomology, principal congruence subgroups and geometric representations

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joint work with
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The braid group

Definition

The $n$-th braid group $B_n$ is the fundamental group of the space of unordered $n$-tuples of distinct points in $\mathbb{C}$.

$$B_n := \pi_1 \left( \mathbb{C}^n \setminus \bigcup_{i<j} \{z_i = z_j\} \bigg/ \mathfrak{S}_n \right)$$
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This is a braid on 4 strands.
The standard presentation of the braid group

The braid group on \( n + 1 \) strands has a presentation given by generators and relations:

\[
\langle \sigma_1, \ldots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle.
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$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| > 1 \rangle.$$

The generator $\sigma_i$ corresponds to the twist

```
\[ \begin{array}{cccc}
| & | & \cdots & | \\
1 & 2 & i & i+1 \\
\end{array} \]
```

\[ \begin{array}{cccc}
| & | & \cdots & | \\
n & n+1 & | & | \\
\end{array} \]
Dehn twist

Let $S$ be an oriented surface and $a$ a simple closed curve in $S$. We call $D_a$ the Dehn twist along $a$.

If two simple curves $a, b$ do not intersect, the corresponding Dehn twists commute $D_a D_b = D_b D_a$. When they intersect in one point, the associated Dehn twists satisfy the braid relation $D_a D_b D_a = D_b D_a D_b$. 
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Let $S_{g,n}$ be an oriented surface of genus $g$, with $n$ boundary components.

**Definition**

We call $\text{MCG}(S_{g,n})$ the mapping class group of $S_{g,n}$, that is the group of isotopy classes of orientation preserving diffeomorphisms of $S_{g,n}$ that fix the boundary pointwise.
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Geometric representation

We can define standard geometric embeddings
\[ \phi : B_{2g+1} \to \text{MCG}(S_{g,1}) \quad \text{and} \quad \phi : B_{2g+2} \to \text{MCG}(S_{g,2}) \]
mapping the standard braid generators to Dehn twist

and hence there is an action on the \( H_1 \) of the surface that preserves the intersection form.

\[ B_{2g+1} \to \text{Aut}(H_1(S_{g,1}; \mathbb{Z}), < \cdot, \cdot >) = \text{Sp}_{2g}(\mathbb{Z}) \]
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Polynomial extension

The previous action naturally extends to the symmetric algebra with $\mathbb{Z}$-linear automorphisms that preserve the degree.

$$H_1(S_g, 1, \mathbb{Z})^* = \langle x_1, y_1, \ldots, x_g, y_g \rangle$$

$$M = \mathbb{Z}[x_1, y_1, \ldots, x_g, y_g]$$

$$B_{2g+1} \rightarrow Aut_\mathbb{Z}(\mathbb{Z}[x_1, y_1, \ldots, x_g, y_g])$$

(and analogous for $B_{2g+2}$). We are interested in the cohomology of braid groups with coefficients in this representation.
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Another point of view (in a special case)

Let $T^2$ be the 2-dimensional compact torus.

**Definition**

$Diff_+(T^2)$ is the group of orientation preserving diffeomorphisms of the torus and $Diff_0 T^2$ is the connected component of the identity.

We have the exact sequence

$$1 \to Diff_0(T^2) \to Diff_+(T^2) \to SL_2(\mathbb{Z}) \to 1$$

that induces the fibration of classifying spaces

$$BDiff_0(T^2) \hookrightarrow BDiff_+(T^2) \to BSL_2(\mathbb{Z})$$
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$$B\text{Diff}_0(T^2) \hookrightarrow B\text{Diff}_+(T^2) \rightarrow B\text{SL}_2(\mathbb{Z})$$
The inclusion $T^2 \hookrightarrow \text{Diff}_0(T^2)$ is an homotopy equivalence.

As a consequence we have the homotopy equivalences

$$B\text{Diff}_0(T^2) \simeq BT^2 \simeq (\mathbb{C}P^\infty)^2$$

and the cohomology of this space is

$$M := H^*(B\text{Diff}_0(T^2); \mathbb{Z}) = \mathbb{Z}[x, y]$$

where $x, y$ are generators in degree 2. We call $M^q$ the homogeneous component of $M$ of degree $q$.

The group $SL_2(\mathbb{Z})$ acts on $\mathbb{Z}[x, y]$ extending the standard action on $\langle x, y \rangle$. 

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The group \( \text{SL}_2(\mathbb{Z}) \) acts on \( \mathbb{Z}[x, y] \) extending the standard action on \( \langle x, y \rangle \).
From the fibration $BDiff_0(T^2) \hookrightarrow BDiff_+(T^2) \to BSL_2(\mathbb{Z})$ we get the Serre spectral sequence

$$E_2^{i,j} = H^i(SL_2(\mathbb{Z}); M^j) \Rightarrow H^{i+j}(BDiff_+(T^2); \mathbb{Z})$$

Theorem

The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product

$\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

Corollary

The spectral sequence above collapses if we tensor the coefficients by a ring $R$ such that 2 and 3 are invertible in $R$. 
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A central extension

There is an extension

$$1 \to \mathbb{Z} \to B_3 \xrightarrow{\psi} SL_2(\mathbb{Z}) \to 1.$$  

defined by $\psi: \sigma_1 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $\psi: \sigma_2 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The kernel of $\psi$ is the index 2 subgroup of the center of $B_3$. The map $\psi$ induces an action of $B_3$ on $(\mathbb{C}P^\infty)^2$. Define the Borel constructions $X := E_{B_3} \times_{B_3} (\mathbb{C}P^\infty)^2$ that fits into the fibration

$$(\mathbb{C}P^\infty)^2 \hookrightarrow X \twoheadrightarrow BB_3.$$  

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\[ (\mathbb{C}P^\infty)^2 \hookrightarrow X \twoheadrightarrow BB_3. \]

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\[ E^{i,j}_2 = H^i(B_3; M^j) \Rightarrow H^{i+j}(X; \mathbb{Z}). \]
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Results

In ’88 Furusawa, Tezuka, Yagita computed the cohomology of $SL_2(\mathbb{Z})$ with coefficients in the module $\mathbb{Q}[x, y]$ and $\mathbb{Z}_p[x, y]$ for any prime $p$.

We compute the cohomology of $SL_2(\mathbb{Z})$ and $B_3$ with coefficients in the module $M = \mathbb{Z}[x, y]$. 
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**Definition**

The *principal congruence subgroup of level* $n$, $\Gamma(n) \subset SL_2(\mathbb{Z})$ is the kernel of the mod-$n$ reduction map

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_n)$$

and the group $B_{\Gamma(n)}$ is the subgroup of $B_3$ that is the counter-image of $\Gamma(n)$ with respect to the projection

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The group $SL_2(\mathbb{Z}_2)$ is the symmetric group $\Sigma_3$ on three elements. The group $B_{\Gamma(2)} \subset B_3$ is the kernel of the map $B_3 \to \Sigma_3$ and hence is the pure braid group $P_3$ on three strands.

By the Kurosh subgroup Theorem, $\Gamma(2) = F_2 \times \mathbb{Z}_2$.
For $n > 2$ the group $\Gamma(n)$ is a free, finitely generated.

We compute the cohomology with coefficients in the module $M = \mathbb{Z}[x, y]$ also for the subgroups $\Gamma(n)$ and $B_{\Gamma(n)}$. 
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Let $k$ be a positive integer and $f$ an holomorphic form on the upper half-plane $\mathbb{H} \cup \{\infty\}$.

**Definition**
The function $f$ is an *cusp integral modular form of weight $k$ (w.r. to $SL_2(\mathbb{Z})$)* if

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z) \quad \forall k \left(\begin{array}{cc}a & b \\c & d\end{array}\right) \in SL_2(\mathbb{Z}).$$

and $f(\infty) = 0$.

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We call $\mathcal{M}_k^0$ the space of cusp modular forms of weight $k$. 
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Filippo Callegaro (Univ. Pisa)
Eicher-Shimura isomorphism

**Theorem**

For $k$ odd the group $H^i(SL_2(\mathbb{Z}); M^{2k} \otimes \mathbb{R})$ is always trivial. For $k$ even we have:

$$H^i(SL_2(\mathbb{Z}); M^{2k} \otimes \mathbb{R}) = \begin{cases} 
\mathcal{M}^0_{k+2} \oplus \mathbb{R} & \text{if } i = 1 \text{ and } k \geq 1 \\
0 & \text{if } i > 0 \text{ or } i = 0 \text{ and } k > 0 \\
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Recall that $M$ is trivial in odd dimension.
Definition (Divided polynomial algebra)

Let $\Delta[x]$ be the sub-$\mathbb{Z}$-module of $\mathbb{Q}[x]$ generated by the elements $x_n := \frac{x^n}{n!}$, for $n \in \mathbb{N}$. For any ring $R$ we define $\Delta_R[x] := \Delta[x] \otimes_{\mathbb{Z}} R$.

The module $\Delta[x]$ is closed by multiplication and the product satisfy the relation $x_i x_j = \left( \binom{i+j}{i} \right) x_{i+j}$. We define in $\Delta[x]$ the ideal $I_p := (p^{v_p(n)+1} x_n, \text{ for } n \in \mathbb{N})$ where $v_p$ is the $p$-adic additive valuation.

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The element $x_{pn}$ generate in $\Delta_p[x]$ a submodule isomorphic to $\mathbb{Z}_{pn+1}$.

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We define also $\Delta^+[x]$ as the submodule of elements with zero constant term.

In more variables we define $\Delta_p[x,y] := \Delta_p[x] \otimes \Delta_p[y]$.

**Theorem**

$$\Delta_p^+[x] = \Delta_{\mathbb{Z}(p)}^+[x]/(px).$$
The element $x_p^n$ generate in $\Delta_p[x]$ a submodule isomorphic to $\mathbb{Z}_{p^{n+1}}$.

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Cohomology of $SL_2(\mathbb{Z})$

**Theorem (-, Cohen, Salvetti)**

\[ H^1(SL_2(\mathbb{Z}); M)(p) = \Delta_p^+ [P_p, Q_p] \]

where $\deg P_p = 2(p + 1)$ and $\deg Q_p = 2p(p - 1)$.

**Theorem (-, C, S)**

For $i > 1$ the cohomology $H^i(SL_2(\mathbb{Z}); M^q)$ is 2-periodic in $i$. The free part is trivial and only 2, 4 and 3 torsion appear. $H^{2i}(SL_2(\mathbb{Z}); M^{8q})$ contains one submodule isomorphic to $\mathbb{Z}_4$. All the others groups are direct sum of modules isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

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Cohomology of $B_3$

**Theorem (-, C, S)**

\[
\begin{align*}
H^1(B_3; M)_p &= H^1(SL_2(\mathbb{Z}); M)_p; \\
H^1(B_3; M^q \otimes \mathbb{Q}) &= H^2(B_3; M^q \otimes \mathbb{Q}) = H^1(SL_2(\mathbb{Z}); M^q \otimes \mathbb{Q}) \text{ for } q > 0; \\
H^2(B_3; M)_p &= H^1(SL_2(\mathbb{Z}); M)_p \text{ for any prime } p \geq 5.
\end{align*}
\]

**Theorem (-, C, S)**

\[
\begin{align*}
H^2(B_3; M)_{(2)} &= \Delta_2^+ [P_2, Q_2] \oplus \mathbb{Z}_2[\overline{Q}_2]/ \sim \\
&\text{with } \frac{Q_2^n}{n!} \sim 2\overline{Q}_2; \\
H^2(B_3; M)_{(3)} &= \Delta_3^+ [P_3, Q_3] \oplus \mathbb{Z}_3[\overline{Q}_3]/ \sim \\
&\text{with } \frac{Q_3^n}{n!} \sim 3\overline{Q}_3 \text{ and } P_3 \frac{Q_3^n}{n!} \sim 0.
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Cohomology of $\Gamma(2)$

The $\Gamma(n)$-invariants in $M$ can be easily computer with a generalization of Dickson invariant theory.

**Theorem (-, C, S)**

Let $F_2$ be the subgroup of $SL_2(\mathbb{Z})$ freely generated by $s_1^2, s_2^2$. The following isomorphisms hold.

For even $n$

\[ H^1(\Gamma(2); M_n) = H^1(F_2; M_n), \]

and for $i > 0$

\[ H^{2i}(\Gamma(2); M_n) = H^0(F_2; M_n \otimes \mathbb{Z}_2) = M_n \otimes \mathbb{Z}_2, \]

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Cohomology of $\Gamma(n)$

Schreier index formula allows to compute the rank of $H^*(\Gamma(m); M_n \otimes \mathbb{Q})$. [Details]

Theorem (-, C, S)

Let $p$ be a prime number and $m > 1$ an integer.

If $p \nmid m$

the $p$-torsion component of $H^1(\Gamma(m); M_n)$ is given by: $H^1(\Gamma(m); M_n)(p) = H^1(SL_2(\mathbb{Z}); M_n)(p) = \Delta^+_p [P_p, Q_p]_{\text{deg}=n}$.

If $p \mid m$, suppose $p^a \mid m$, $p^{a+1} \nmid m$. Then we have

$H^1(\Gamma(m); M_{>0})(p) \simeq \Delta^+_p [x, y]$, where $x, y$ have degree 1.
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\]
Theorem (-, C, S)

\[ H^0(B_{\Gamma(2)}; M_0) = \mathbb{Z}, \ H^1(B_{\Gamma(2)}; M_0) = \mathbb{Z}^3, \ H^2(B_{\Gamma(2)}; M_0) = \mathbb{Z}^2 \]

Let \( n > 0 \); for even \( n \);
\[ H^0(B_{\Gamma(2)}; M_n) = H^0(\Gamma(2); M_n), \]
\[ H^1(B_{\Gamma(2)}; M_n) = H^2(B_{\Gamma(2)}; M_n) = H^1(\Gamma(2); M_n) \]
for odd \( n \);
\[ H^0(B_{\Gamma(2)}; M_n) = 0, \]
\[ H^1(B_{\Gamma(2)}; M_n) = H^1(\Gamma(2); M_n) = M_n \otimes \mathbb{Z}_2, \]
\[ H^2(B_{\Gamma(2)}; M_n) = H^2(\Gamma(2); M_n) = (M_n \oplus M_n) \otimes \mathbb{Z}_2 \]

for any \( m > 2 \), for any \( n \):
\[ H^*(B_{\Gamma(m)}; M_n) = H^*(\Gamma(m); M_n) \otimes H^*(\mathbb{Z}; \mathbb{Z}). \]
Cohomology of $B_{\Gamma(n)}$

**Theorem (-, C, S)**

\[
H^0(B_{\Gamma(2)}; M_0) = \mathbb{Z}, \quad H^1(B_{\Gamma(2)}; M_0) = \mathbb{Z}^3, \quad H^2(B_{\Gamma(2)}; M_0) = \mathbb{Z}^2
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for odd $n$:

\[
H^0(B_{\Gamma(2)}; M_n) = 0, \\
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\]
### Theorem (\(-, C, S\))

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\begin{align*}
H^0(B_{\Gamma(2)}; M_0) &= \mathbb{Z}, \quad H^1(B_{\Gamma(2)}; M_0) = \mathbb{Z}^3, \quad H^2(B_{\Gamma(2)}; M_0) = \mathbb{Z}^2 \\
\text{Let } n &> 0; \text{ for even } n; \quad H^0(B_{\Gamma(2)}; M_n) = H^0(\Gamma(2); M_n), \\
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\text{for odd } n; \quad H^0(B_{\Gamma(2)}; M_n) = 0, \\
H^1(B_{\Gamma(2)}; M_n) &= H^1(\Gamma(2); M_n) = M_n \otimes \mathbb{Z}_2, \\
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\text{for any } m > 2, \text{ for any } n: \\
H^*(B_{\Gamma(m)}; M_n) &= H^*(\Gamma(m); M_n) \otimes H^*(\mathbb{Z}; \mathbb{Z}).
\end{align*}
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Cohomology of $B_{\Gamma(n)}$

**Theorem** $(-, C, S)$

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for odd $n$: $H^0(B_{\Gamma(2)}; M_n) = 0$,

$H^1(B_{\Gamma(2)}; M_n) = H^1(\Gamma(2); M_n) = M_n \otimes \mathbb{Z}_2$,

$H^2(B_{\Gamma(2)}; M_n) = H^2(\Gamma(2); M_n) = (M_n \oplus M_n) \otimes \mathbb{Z}_2$

for any $m > 2$, for any $n$:

$H^*(B_{\Gamma(m)}; M_n) = H^*(\Gamma(m); M_n) \otimes H^*(\mathbb{Z}; \mathbb{Z})$. 

Filippo Callegaro (Univ. Pisa)
Methods

- $SL_2(\mathbb{Z}) = \mathbb{Z}_4 \ast_{\mathbb{Z}_2} \mathbb{Z}_6$;
- Dickson’s invariant theory for $SL_2(\mathbb{Z})$;
- explicit computations for $H^*(G; M)$, $G = \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6$;
- study of the spectral sequence for $\mathbb{Z} \to B_3 \to SL_2(\mathbb{Z})$;
- study of the maps of spectral sequences induced by $\mathbb{Z}_4 \to SL_2(\mathbb{Z})$ and $\mathbb{Z}_6 \to SL_2(\mathbb{Z})$;
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In ’79 Cohen, Moore and Neisendorfer constructed a family of maps

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such that the composition $\Omega^2 S^{2n+1} \xrightarrow{\alpha_n} S^{2n-1} \xrightarrow{E} \Omega^2 S^{2n+1}$ with the double suspension gives, up to homotopy, the $p^r$ power map, for any prime $p \geq 3$, and $r \geq 1$.

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Anick fibration

Cohen, Moore and Neisendorfer conjectured the existence of a $p$-local fibration

$$S^{2n-1} \to T_{p^r}(2n + 1) \to \Omega S^{2n+1}.$$ 

with connecting map $\Omega^2 S^{2n+1} \xrightarrow{\alpha_n} S^{2n-1}$.

In '93 Anick constructed such a fibration sequence for $p > 3$. In 2007 Gary and Theriault gave a construction that is valid also for $p = 3$.

**Theorem**

The reduced cohomology of the space $T_p(2n + 1)$ is given by:

$$\overline{H}^i(T_p(2n + 1); \mathbb{Z}_p) = \begin{cases} \mathbb{Z}/p^r & \text{if } i = 2np^{r-1}k, p \nmid k; \\ 0 & \text{otherwise.} \end{cases}$$
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Let $p \geq 5$ be a prime.

a) $H^\ast(EB_3 \times B_3 (\mathbb{C}P^\infty)^2; \mathbb{Z})_{(p)} = H^\ast(S^1 \times BDiff_+ (T^2); \mathbb{Z})_{(p)}$

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**Theorem (\(-,C, S\))**

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\[ \Sigma^2 \left(T_p(2p + 3) \times T_p(2p^2 - 2p + 1)\right) \]
A surprising relation

Theorem $(-, C, S)$

Let $p \geq 5$ be a prime.

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b) The $p$-torsion component in the cohomology group $H^*(EB_3 \times B_3 (\mathbb{C}P^\infty)^2; \mathbb{Z})$ is isomorphic to the reduced cohomology of the space $\Sigma^2(T_p(2p + 3) \times T_p(2p^2 - 2p + 1)) \vee \Sigma(T_p(2p + 3) \vee T_p(2p^2 - 2p + 1))$. 

Question

Is there any topological explanation for the isomorphism above?
A surprising relation

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Let \( p \geq 5 \) be a prime.

a) \( H^*(EB_3 \times B_3 (\mathbb{C}P^\infty)^2; \mathbb{Z}) (p) = H^*(S^1 \times BDiff_+ (T^2); \mathbb{Z}) (p) \)

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\Sigma^2 (T_p(2p+3) \times T_p(2p^2 - 2p + 1)) \vee \Sigma (T_p(2p+3) \vee T_p(2p^2 - 2p + 1)).
\]

**Question**

Is there any topological explanation for the isomorphism above?
Thank you for your attention!
Theorem

The group $H^0(\Gamma(m); M_n)$ is isomorphic to $M_0$ for $n = 0$ and is trivial for $n > 0$.

Theorem

Let $m > 2$ be an integer that factors as $m = p_1^{a_1} \cdots p_k^{a_k}$. The cardinality of $SL_2(\mathbb{Z}_m)$ is given by $d = \prod_i p_i^{(a_i-1)3} p_i(p_i^2 - 1)$ and if we define $i = \frac{d}{p_1(p_1^2 - 1)}$ then $\Gamma(m)$ is a free group of rank

$$r = \begin{cases} 
\frac{i}{2} + 1 & \text{if } p_1 = 2 \\
i(p_1(p_1^2 - 1) - 1) + 1 & \text{if } p_1 > 2 
\end{cases}$$

The rank of the group $H^1(\Gamma(m); M_n \otimes \mathbb{Q})$ is $r$, for $n = 0$ and $(r - 1)(n + 1)$ for $n > 0$. 

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