Combinatorics and essential coordinate components of characteristic varieties of line arrangements

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- $\mathcal{B} \subset \mathcal{A}$ sub-arrangement:

$$\begin{array}{ccc} \mathbb{T}_{G_{\mathcal{B}}} & \hookrightarrow & \mathbb{T}_G \\ \downarrow & & \downarrow \\ \{t_{\bar{L}} = 1,\ \bar{L} \in \mathcal{A} \setminus \mathcal{B}\} = (\mathbb{C}^*)^{\mathcal{B}} & \hookrightarrow & (\mathbb{C}^*)^{\mathcal{A}} \end{array}$$

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Definition

Characteristic varieties of $\mathcal{A}$:

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- Determined by Betti numbers of finite cyclic covers.

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Theorem (Libgober)

1. *There is an algorithm to compute non sub-essential coordinate components which depends on the position of multiple points of $\mathcal{A}$.*
2. *Sub-essential coordinate components are torsion points.*

Coverings and torsion characters

- $\xi \in \mathbb{T}_G$, $1 < \operatorname{ord}(\xi) =: N < \infty$, $\xi(\mu_j) = \exp\left(2i\pi \frac{k_j}{N}\right)$,

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- $\tilde{\xi} : X_\xi \to X_\xi$ generator of the monodromy group of ρ.

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- $H^1(X_\xi; \mathbb{C})^\xi$ eigenspace for $\tilde{\xi}$ and eigenvalue $\exp(\frac{2i\pi}{N})$.
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- $\{\xi\}$ sub-essential coordinate $\Longrightarrow$

  $$H^1(\overline{X}_\xi; \mathbb{C}) \neq H^1(X_\xi; \mathbb{C}) \Longleftrightarrow H^0(\overline{X}_\xi; \Omega^1_{\overline{X}_\xi}) \neq H^0(\overline{X}_\xi; \Omega^1_{\overline{X}_\xi}\log(D_\xi))$$

An explicit compactification

Notation

- $\mathcal{P} := \{\bar{L}_i \cap \bar{L}_j \mid 0 \leq i < j \leq n\}$.
- For $P \in \mathcal{P}$, $m(P) := \#\{\bar{L} \in \mathcal{A} \mid P \in \bar{L}\}$.
- $\mathcal{P}_{>2} := \{P \in \mathcal{P} \mid m(P) > 2\}$.

An explicit compactification

First Step

- Blow up the points of $\mathcal{P}_{>2}$, $\sigma : Y \to \mathbb{P}^2$.
- $\tilde{\mathcal{A}}$ irreducible components of $\sigma^{-1}(\bigcup \mathcal{A})$, normal crossing divisor:
    - $\tilde{L}$ strict transform of $\bar{L}$, $\tilde{L}^2 = 1 - \#\mathcal{P}_{>2} \cap \bar{L}$.
    - $E_P := \sigma^{-1}(P)$, $P \in \mathcal{P}_{>2}$, exceptional component, $E_P^2 = -1$, $\mu_P := \prod_{P \in \bar{L}} \mu_{\bar{L}}, \quad t_{E_P} = t_P := \xi(\mu_P)$.

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Second Step

$$\tilde{Z}_\xi := \left\{ [x : y : z : T]_{(1,1,1,k)} \in \mathbb{P}^3_{(1,1,1,k)} \;\middle|\; T^N = \prod_{j=0}^{n} \ell_j(x,y,z)^{k_j} \right\}$$

$$\begin{array}{ccccccc} \overline{X}_\xi & \xrightarrow{\pi} & \tilde{Y}_\xi^{\text{norm}} & \longrightarrow & \tilde{Y}_\xi & \longrightarrow & \tilde{Z}_\xi \\ & & & & \downarrow & & \downarrow \\ & & & & Y & \longrightarrow & \mathbb{P}^2 \end{array}$$

Exact sequences

- $0 \to \Omega^1_{X_\xi} \to \Omega^1_{X_\xi} \log(D_\xi) \to \bigoplus_{D \in \mathcal{D}_\xi} i_* \mathcal{O}_D \to 0$

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- $\mathbb{C}\langle \mathcal{B}_\xi(B) \rangle^\xi = \begin{cases} 0 & \text{if } \#\mathcal{B}_\xi(B) < N \\ \mathbb{C}\left\langle \overbrace{\sum_{j=0}^{N-1} \exp\left(-2i\pi \frac{j}{N}\right) \xi^j(\tilde{B})}^{B(\xi)} \right\rangle & \text{if } \#\mathcal{B}_\xi(B) = N \end{cases}$

Inner components

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B is *inner* if $\#\mathcal{B}_\xi(B) = N \iff t_B = 1$ and $t_C = 1$ for all C neighbor to B: $\mathcal{U}_\xi$ set of inner components.

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2. $\dim_{\mathbb{C}} K_\xi$ *is the corank of the intersection form induced on* $\bigoplus_{B\in\mathcal{U}_\xi} \mathbb{C}\langle B(\xi)\rangle$ *by the usual intersection form of* $\overline{X}_\xi$.

Twisted intersection form

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- If $\Gamma_{\mathcal{U}_\xi}$ has cycles, it depends on the embedding of the boundary of a regular neighborhood of $\bigcup\mathcal{A}$ in $M(\mathcal{A})$ (Hironaka, Florens-Marco-Guerville)

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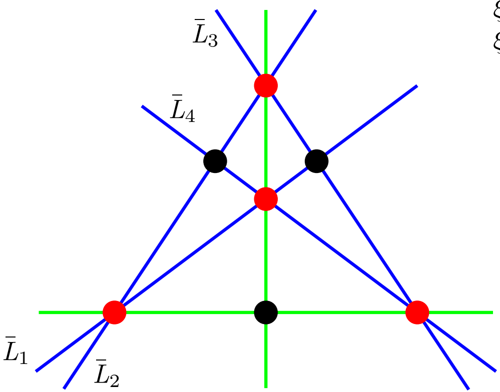
- $B \cdot_\xi C = \begin{cases} 0 & \text{if } B\cdot C = 0 \\ B\cdot C & \text{if either } B = C \text{ or } \overrightarrow{BC}\subset \mathcal{T}_{\mathcal{U}_\xi} \\ \xi(\gamma_{B,C})(B\cdot C) & \text{if } \overrightarrow{BC}\not\subset\mathcal{T}_{\mathcal{U}_\xi}\end{cases}$

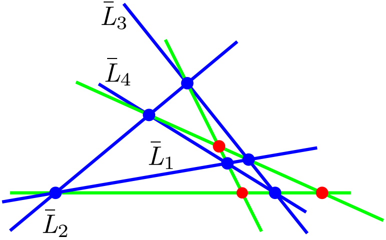
Ceva arrangement



$$\xi(\mu_1) = \zeta_d$$
$$\xi(\mu_2) = \zeta_d^{-1}$$
$$\xi(\mu_3) = \zeta_d$$
$$\xi(\mu_4) = \zeta_d^{-1}$$

$$\cdot_\xi : \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Ceva-extended arrangement



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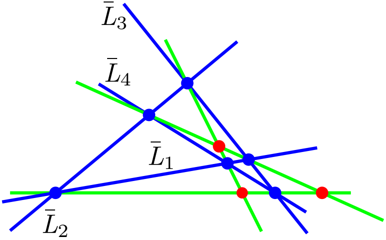

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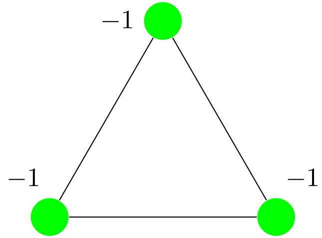

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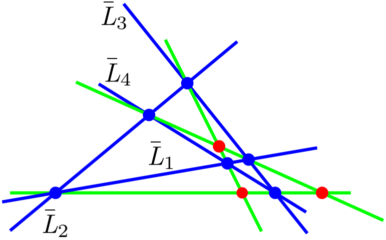

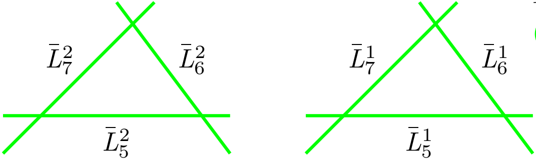

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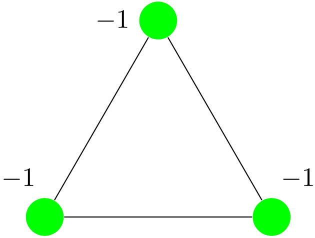
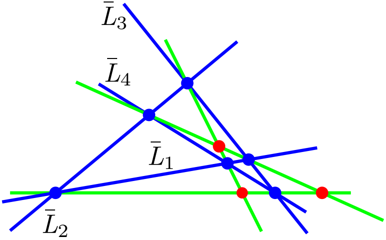
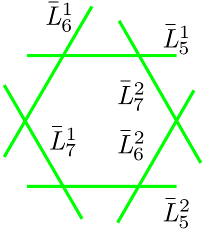


$$\cdot_\xi : \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Ceva-extended arrangement





Cohen-Suciu

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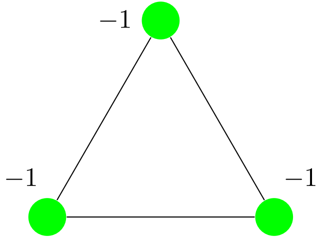

$$\cdot_\xi : \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

Hesse arrangement

$$0 = xyz\left((x^3 + y^3 + z^3)^3 - 27xyz\right)$$

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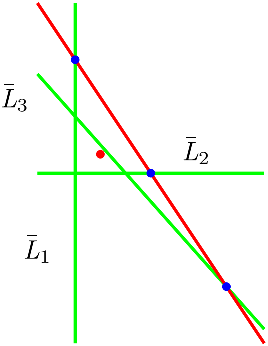
$$0 = xyz(x^3+y^3+z^3-3xyz)(\dots)$$



$$\xi(\mu_1)=\xi(\mu_2)=\xi(\mu_3)=1$$
$$\xi(\mu_4)=\xi(\mu_5)=\xi(\mu_6)=t\in\mathbb{C}^*$$
$$\xi(\mu_7)=\xi(\mu_8)=\xi(\mu_9)=s\in\mathbb{C}^*$$
$$\xi(\mu_{10})=\xi(\mu_{11})=\xi(\mu_{12})=u\in\mathbb{C}^*$$
$$tsu=1$$

$$\cdot_\xi : \begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}$$

Mulţimesc!