

Handout - Derivative - Chain Rule

Power-Chain Rule a, b are constants.

Function	Derivative	
$y = a \cdot x^n$	$\frac{dy}{dx} = a \cdot n \cdot x^{n-1}$	Power Rule
$y = a \cdot u^n$	$\frac{dy}{dx} = a \cdot n \cdot u^{n-1} \cdot \frac{du}{dx}$	Power-Chain Rule

Ex1a. Find the derivative of $y = 8(6x + 21)^8$

$$\begin{aligned}\text{Answer: } y' &= \mathbf{384(6x + 21)^7} \\ a &= 8, \quad n = 8 \\ u &= 6x + 21 \quad \Rightarrow \frac{du}{dx} = 6 \\ \Rightarrow y' &= 8 \cdot 8 \cdot (6x + 21)^7 \cdot 6\end{aligned}$$

Ex1b. Find the derivative of $y = 8(4x^2 + 7x + 28)^4$

$$\begin{aligned}\text{Answer: } y' &= \mathbf{32(8x + 7)(4x^2 + 7x + 28)^3} \\ a &= 8, \quad n = 4 \\ u &= 4x^2 + 7x + 28 \quad \Rightarrow \frac{du}{dx} = 8x + 7 \\ \Rightarrow y' &= 8 \cdot 4 \cdot (4x^2 + 7x + 28)^3 \cdot (8x + 7)\end{aligned}$$

Ex1c. Find the derivative of $y = 2\sqrt{6x^2 + 4x + 26}$

$$\begin{aligned}\text{Answer: } y' &= \frac{\mathbf{12x + 4}}{\sqrt{\mathbf{6x^2 + 4x + 26}}} \\ a &= 2, \quad n = \frac{1}{2} \\ u &= 6x^2 + 4x + 26 \quad \Rightarrow \frac{du}{dx} = 12x + 4 \\ \Rightarrow y' &= 2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{6x^2 + 4x + 26}} \cdot (12x + 4)\end{aligned}$$

Exercises

Find the derivatives of the expressions

a) $5(9x + 25)^8$

b) $7(2x + 24)^8$

c) $2(4x^2 + 4x + 21)^9$

d) $6(7x^2 + 4x + 22)^4$

e) $7(7x^2 + 9x + 24)^{13/3}$

f) $3(7x^2 + 4x + 29)^{22/3}$

g) $5\sqrt{x^2 + 8x + 25}$

h) $7\sqrt{7x^2 + 3x + 24}$

i) $\frac{8}{\sqrt{3x^2+3x+22}}$

j) $\frac{3}{\sqrt{2x^2+2x+22}}$

Answers a) $360(9x + 25)^7$; b) $112(2x + 24)^7$;

c) $18(8x + 4)(4x^2 + 4x + 21)^8$; d) $24(14x + 4)(7x^2 + 4x + 22)^3$;

e) $\frac{91}{3}(14x + 9)(7x^2 + 9x + 24)^{10/3}$; f) $22(14x + 4)(7x^2 + 4x + 29)^{19/3}$;

g) $\frac{5(2x+8)}{2\sqrt{x^2+8x+25}}$; h) $\frac{7(14x+3)}{2\sqrt{7x^2+3x+24}}$;

i) $-\frac{4(6x+3)}{(3x^2+3x+22)^{3/2}}$; j) $-\frac{3(4x+2)}{2(2x^2+2x+22)^{3/2}}$;

Sine and Cosine - Chain Rules a, b are constants.

Function	Derivative	
$y = \sin(x)$	$\frac{dy}{dx} = \cos(x)$	Sine Rule
$y = \cos(x)$	$\frac{dy}{dx} = -\sin(x)$	Cosine Rule
$y = a \cdot \sin(u)$	$\frac{dy}{dx} = a \cdot \cos(u) \cdot \frac{du}{dx}$	Chain-Sine Rule
$y = a \cdot \cos(u)$	$\frac{dy}{dx} = -a \cdot \sin(u) \cdot \frac{du}{dx}$	Chain-Cosine Rule

Ex2a. Find $\frac{dy}{dx}$ where $y = 2 \sin(9x^3 + 3x^2 + 1)$

<p>Answer: $2(27x^2 + 6x) \cos(9x^3 + 3x^2 + 1)$ $a = 2$ $u = 9x^3 + 3x^2 + 1 \Rightarrow \frac{du}{dx} = 27x^2 + 6x$</p>
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Ex2b. Find $\frac{dy}{dx}$ where $y = 5 \cos(9x^5 + 5x^4 + 3)$

<p>Answer: $-5(45x^4 + 20x^3) \sin(9x^5 + 5x^4 + 3)$ $a = 5$ $u = 9x^5 + 5x^4 + 3 \Rightarrow \frac{du}{dx} = 45x^4 + 20x^3$</p>

Ex2c. Find the derivative of $y = (4 \sin(4x) + 4 \cos(5x))^4$

<p>Answer: $4 \cdot (16 \cos(4x) - 20 \sin(5x)) \cdot (4 \sin(4x) + 4 \cos(5x))^3$ $n = 4$ $u = 4 \sin(4x) + 4 \cos(5x)$ $\Rightarrow \frac{du}{dx} = 16 \cos(4x) - 20 \sin(5x)$</p>

Exercises

Find the derivatives of the expressions

a) $7 \sin(7x^3 + 7x^2 + 8)$

b) $5 \cos(2x^4 + 7x^3 + 9)$

c) $8 \sin(8x^3 + 9x^2 + 1)$

d) $5 \cos(6x^4 + 8x^3 + 2)$

e) $(\sin(x) + 9 \cos(x^4))^4$

f) $(2 \sin(6x) + 9 \cos(3x^2))^3$

g) $(4 \sin(5x^3) + 8 \cos(8x^5))^5$

h) $(\sin(7x^3) + 2 \cos(6x^4))^3$

i) $(5 \sin(5x^5) + 5 \cos(7x^4))^{\frac{1}{2}}$

j) $(6 \sin(2x^3) + 5 \cos(8x^2))^{\frac{3}{2}}$

Answers a) $7(21x^2 + 14x) \cos(7x^3 + 7x^2 + 8)$; b) $-5(8x^3 + 21x^2) \sin(2x^4 + 7x^3 + 9)$;
c) $8(24x^2 + 18x) \cos(8x^3 + 9x^2 + 1)$; d) $-5(24x^3 + 24x^2) \sin(6x^4 + 8x^3 + 2)$;
e) $4 \cdot (\cos(x) - 36x^3 \sin(x^4)) \cdot (\sin(x) + 9 \cos(x^4))^3$;
f) $3 \cdot (12 \cos(6x) - 54x \sin(3x^2)) \cdot (2 \sin(6x) + 9 \cos(3x^2))^2$;
g) $5 \cdot (60x^2 \cos(5x^3) - 320x^4 \sin(8x^5)) \cdot (4 \sin(5x^3) + 8 \cos(8x^5))^4$;
h) $3 \cdot (21x^2 \cos(7x^3) - 48x^3 \sin(6x^4)) \cdot (\sin(7x^3) + 2 \cos(6x^4))^2$;
i) $\frac{1}{2} \cdot (125x^4 \cos(5x^5) - 140x^3 \sin(7x^4)) \cdot (5 \sin(5x^5) + 5 \cos(7x^4))^{-\frac{1}{2}}$;
j) $\frac{3}{2} \cdot (36x^2 \cos(2x^3) - 80x \sin(8x^2)) \cdot (6 \sin(2x^3) + 5 \cos(8x^2))^{\frac{1}{2}}$;

Exponent and Logarithmic - Chain Rules a, b are constants.

Function	Derivative	
$y = e^x$	$\frac{dy}{dx} = e^x$	Exponential Function Rule
$y = \ln(x)$	$\frac{dy}{dx} = \frac{1}{x}$	Logarithmic Function Rule
$y = a \cdot e^u$	$\frac{dy}{dx} = a \cdot e^u \cdot \frac{du}{dx}$	Chain-Exponent Rule
$y = a \cdot \ln(u)$	$\frac{dy}{dx} = \frac{a}{u} \cdot \frac{du}{dx}$	Chain-Log Rule

Ex3a. Find the derivative of $y = 6e^{7x+22}$

Answer: $y' = 42e^{7x+22}$ $a = 6$ $u = 7x + 22 \Rightarrow \frac{du}{dx} = 7 \Rightarrow y' = 6 \cdot e^{7x+22} \cdot 7$

Ex3b. Find the derivative of $y = 6e^{7x^2+3x+22}$

Answer: $y' = 6 \cdot (14x + 3) \cdot e^{7x^2+3x+22}$ $a = 6$ $u = 7x^2 + 3x + 22 \Rightarrow \frac{du}{dx} = 14x + 3 \Rightarrow y' = 6 \cdot e^{7x^2+3x+22} \cdot (14x + 3)$
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Ex3c. Find the derivative of $y = -2e^{8x^{8/3}+7x^{7/2}}$

Answer: $y' = -2 \cdot \left(\frac{49x^{5/2}}{2} + \frac{64x^{5/3}}{3} \right) \cdot e^{8x^{8/3}+7x^{7/2}}$ $a = -2$ $u = 8x^{8/3} + 7x^{7/2} \Rightarrow \frac{du}{dx} = \frac{49x^{5/2}}{2} + \frac{64x^{5/3}}{3}$ $\Rightarrow y' = -2 \cdot e^{8x^{8/3}+7x^{7/2}} \cdot \left(\frac{49x^{5/2}}{2} + \frac{64x^{5/3}}{3} \right)$

Ex3d. Find the derivative of $y = 7 \ln(3x + 26)$

$$\begin{aligned} \text{Answer: } y' &= \frac{21}{3x + 26} \\ a &= 7 \\ u &= 3x + 26 \Rightarrow \frac{du}{dx} = 3 \Rightarrow y' = \frac{7}{3x+26} \cdot 3 \end{aligned}$$

Ex3e. Find the derivative of $y = 8 \ln(2x^2 + 9x + 26)$

$$\begin{aligned} \text{Answer: } y' &= \frac{8(4x + 9)}{2x^2 + 9x + 26} \\ a &= 8 \\ u &= 2x^2 + 9x + 26 \Rightarrow \frac{du}{dx} = 4x + 9 \\ &\Rightarrow y' = \frac{8}{2x^2+9x+26} \cdot (4x + 9) \end{aligned}$$

Ex3f. Find the derivative of $y = 7 \log(9x^{5/2} + 5x^{5/3})$

$$\begin{aligned} \text{Answer: } y' &= 7 \cdot \left(\frac{45x^{3/2}}{2} + \frac{25x^{2/3}}{3} \right) \cdot \frac{1}{9x^{5/2} + 5x^{5/3}} \\ a &= 7 \\ u &= 9x^{5/2} + 5x^{5/3} \Rightarrow \frac{du}{dx} = \frac{45x^{3/2}}{2} + \frac{25x^{2/3}}{3} \\ &\Rightarrow y' = 7 \cdot e^{9x^{5/2}+5x^{5/3}} \cdot \left(\frac{45x^{3/2}}{2} + \frac{25x^{2/3}}{3} \right) \end{aligned}$$

Exercises

Find the derivatives of the expressions

a) $6e^{8x+26}$

b) $6e^{6x+21}$

c) $4 \ln(2x + 21)$

d) $6 \ln(3x + 27)$

e) $6e^{3x^2+8x+27}$

f) $8e^{4x^2+6x+29}$

g) $4 \ln(4x^2 + 8x + 29)$

h) $6 \ln(7x^2 + 3x + 25)$

i) $6e^{5x^{8/3}+5\sqrt{x}}$

j) $9 \ln(9x^{8/3} + 8\sqrt{x})$

Answers a) $48e^{8x+26}$; b) $36e^{6x+21}$; c) $\frac{8}{2x+21}$; d) $\frac{18}{3x+27}$;

e) $6 \cdot (6x + 8) \cdot e^{3x^2+8x+27}$; f) $8 \cdot (8x + 6) \cdot e^{4x^2+6x+29}$;

g) $\frac{4(8x+8)}{4x^2+8x+29}$; h) $\frac{6(14x+3)}{7x^2+3x+25}$;

i) $6 \cdot \left(\frac{40x^{5/3}}{3} + \frac{5}{2\sqrt{x}}\right) \cdot e^{5x^{8/3}+5\sqrt{x}}$; j) $9 \cdot \left(24x^{5/3} + \frac{4}{\sqrt{x}}\right) \cdot \frac{1}{9x^{8/3}+8\sqrt{x}}$;

Exercises

Find the derivatives of the expressions

a) $5 \sin(5x) + 4e^{4x+1} + \frac{4}{\sqrt{x}}$

b) $2 \cos(5x) + 5e^{3x+1} + \frac{8}{\sqrt{x-1}}$

c) $2 \cos(4x) + 4 \ln(3x - 1) - \frac{7}{(3x+1)^{3/5}}$

d) $\cos(5x) + 5e^{x+1} - 7(x - 1)^{2/3}$

e) $\left(3 \cos(5x) + 2 \ln(4x) + 8\sqrt{x}\right)^5$

f) $\left(\cos(4x) + 3 \ln(3x) - 7\sqrt[5]{x}\right)^3$

g) $\left(4 \sin(3x) + 3 \ln(x) - \frac{6}{\sqrt{x}}\right)^{-\frac{1}{2}}$

h) $\left(4 \sin(5x) + e^{3x} - \frac{4}{\sqrt[4]{x}}\right)^{\frac{5}{2}}$

Answers a) $25 \cos(5x) + 16e^{4x+1} - \frac{2}{x^{3/2}}$; b) $-10 \sin(5x) + 15e^{3x+1} - \frac{4}{(x-1)^{3/2}}$;

c) $-8 \sin(4x) + \frac{12}{3x-1} + \frac{63}{5(3x+1)^{8/5}}$; d) $-5 \sin(5x) + 5e^{x+1} - \frac{14}{3\sqrt[3]{x-1}}$;

e) $5 \cdot \left(-15 \sin(5x) + \frac{2}{x} + \frac{4}{\sqrt{x}}\right) \cdot \left(3 \cos(5x) + 2 \ln(4x) + 8\sqrt{x}\right)^4$;

f) $3 \cdot \left(-4 \sin(4x) + \frac{3}{x} - \frac{7}{5x^{4/5}}\right) \cdot \left(\cos(4x) + 3 \ln(3x) - 7\sqrt[5]{x}\right)^2$;

g) $-\frac{1}{2} \cdot \left(12 \cos(3x) + \frac{3}{x} + \frac{3}{x^{3/2}}\right) \cdot \left(4 \sin(3x) + 3 \ln(x) - \frac{6}{\sqrt{x}}\right)^{-\frac{3}{2}}$;

h) $\frac{5}{2} \cdot \left(20 \cos(5x) + 3e^{3x} + \frac{1}{x^{5/4}}\right) \cdot \left(4 \sin(5x) + e^{3x} - \frac{4}{\sqrt[4]{x}}\right)^{\frac{3}{2}}$;